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RECENT RESEARCHES

IN THE

THEORY OF FOURIER SERIES

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Allen A. Shaw

SIX LECTURES

ON

RECENT RESEARCHES IN THE THEORY OF FOURIER SERIES

BY

GANESH PRASAD

HARDINGE PROFESSOR OF HIGHER MATHEMATICS

IN THE CALCUTTA UNIVERSITY

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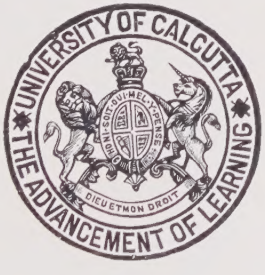

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PREFACE

In this book are reproduced without any material alteration the lectures, which I delivered last winter with the aim of placing before my audience, in as compact and simple a form as possible, the *present* state of the convergence problem in the theory of Fourier series. By the publication of this book, the lectures will become accessible to a larger public, and it is hoped that they will be found useful by all advanced students of Higher Pure Mathematics.

In addition to a large number of original papers, quoted in the book, publications of a different character have also been of much help to me; I may mention specially the article of Hilb and M. Riesz in Vol. II C of the *Encyklopädie der Mathematischen Wissenschaften* and the accounts of Fourier series in Schlesinger and Plessner's *Lebesguesche Integrale und Fouriersche Reihen*, de la Vallée Poussin's *Cours d'analyse infinitésimale*, t. 2, and Hobson's *Theory of Functions of a Real Variable*, Vol. 2. In the case of every theorem or criterion, I have tried to give the simplest proof, and in many cases the original proofs of the discoverers.

A few words will not be out of place here in reference to my criticism of (1) Lebesgue's criterion for summability (C1), (2) his criterion for summability (C2) and (3) the criteria of Carleman and Hardy and Littlewood for strong summability (C1). My criticism has been before experts for some months past and I have reason to believe that they think I am right. It seems that too little attention has been paid by previous writers to the question of the differentiability of an indefinite integral at *a given point*; perhaps the only exception among them is the outstanding personality of Leopold Fejér, who took up a critical attitude towards Lebesgue's criterion for summability (C1) in a short paper * which appeared in *Göttinger Nachrichten* in 1926. It was a serious loss to the cause of truth that so eminent an authority as Fejér did not condescend to go into the question fully and with care. My own research work in connection with the various criteria was planned out as early as 1924; and after dealing, in a number of papers, published in 1924-1927, with the question of differentiability, I began to publish my results about the criteria last winter; these results will be found in the March, June and September issues of Vol. XIX of the *Bulletin of the Calcutta Mathematical Society*.

55/36-G

* "Ueber die arithmetischen Mittel erster Ordnung der Fourierreihe."

The table of contents is intentionally given in a detailed form, and its length as well as the printing of the names of the authors of important results in thick black type is expected to make up for the absence of indices of names and subjects.

In the reading of the proof-sheets I have been helped by my pupils, Mr. Hariprasanna Banerji, M.Sc., Lecturer to Postgraduate students in Pure Mathematics in the Calcutta University, and Mr. Badri Nath Prasad, M.Sc., Lecturer in the Department of Mathematics in the Allahabad University ; I take his opportunity to express my thanks to both of them.

CALCUTTA, GANESH PRASAD.

August 14, 1928.

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FIRST LECTURE.

Introductory.

Colleagues, students and other gentlemen!

The six lectures which I propose to deliver on the present state of the convergence problem in the theory of Fourier's series may be considered to be a review of nearly a thousand publications on Fourier's series which have appeared in the last twenty-seven years in various countries and in various languages. Although I realize the difficult nature of the task before me, I shall feel amply remunerated if I succeed in inspiring some of you with enthusiasm for research in the subject of Fourier's series which has always had a fascination for most of the leading mathematicians of the last 150 years.

§ 1.

1. To-day's lecture is intended to be of a historical and introductory character, and I shall begin it by giving you an idea of the state of the convergence problem in the theory of Fourier's series at the end of the last century. At the end of 1900, our stock of knowledge about that problem included a number of conditions each sufficient for the convergence of Fourier's series at a point; a few conditions sufficient for its uniform convergence in an interval; Du Bois-Reymond's results, including, (*a*) an example of a continuous function with divergent Fourier's series at a point, (*b*) an example of a continuous function with divergent Fourier's series at the points of an everywhere dense set and (*c*) the behaviour of the Fourier's series of a function $f(x)$ at a point x_0, where the function has a discontinuity of the second kind so that

$$f(x_0+t)+f(x_0-t)=\chi(t)\cos\psi(t),$$

χ and ψ being both monotone and ψ having infinity as its limit for t tending to 0; Cantor's theorem on the uniqueness of the representation of a function by Fourier's series; and the summation methods of Poisson and Riemann. Beyond Parseval's theorem, and Riemann's result about the vanishing of the Fourier coefficients of an absolutely integrable function with $\frac{1}{n}$, not much might be said to have been known about the properties

of Fourier coefficients. Also Cesàro summation of Fourier's series and Fejér's theorem became known in the beginning of the present century which also saw the introduction of the notion of "strong" summability by Hardy and Littlewood, and more persistent attempts by Lusin, Neder, Steinhaus and Kolmogoroff at answering Du Bois-Reymond's question: Can a continuous function be found for which the Fourier's series diverges at *every* point? That the work achieved by the end of the last century was creditable, will be easily admitted when I say that, before Dirichlet gave the first rigorous treatment of the convergence problem in 1829, in spite of the efforts of Fourier, Poisson and Cauchy,*

* Sachse in his "*Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variabeln durch trigonometrische Reihen*" (Gœttingen, 1879) says: "Fourier did not prove that the trigonometric series which he obtained for the representation of the function really converges to the value of the function. In truth, he gives a rigorous definition of the convergence of a series (*Théorie de la Chaleur*, Art. 177); but he regards in the actual case, the demonstration of the convergence to be easy, and leaves it to the reader to find it. In the particular cases which one has throughout to consider and for the particular values of the variable, one recognises that the series converges to the value of the function, but Cauchy appears to have been the first who felt the necessity of a rigorous demonstration, independent of the form of the function and applicable to all the values of the variable. He communicated in 1826 to the Academy of Sciences a demonstration of the formula of Fourier. He brings the general term of the series to a form which proves that the ratio of that term to $A\frac{\sin nx}{n}$, where A is a constant independent of n, differs less and less from unity as n is made larger and larger. From the fact that the series of which the general term is $A\frac{\sin nx}{n}$ is convergent, Cauchy concludes that the same is the case for the series of Fourier." Dirichlet pointed out the defect in Cauchy's reasoning by citing the cases of $\sum_1^\infty \frac{(-1)^n}{\sqrt{n}}$ and $\sum_1^\infty \left\{\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}\right\}$ which have general terms tending to equality with increasing n but are, notwithstanding this, not both convergent, the first series being convergent and the second divergent. Cauchy's proof is based on his theorem of residues and contains another defect on account of the supposition that $f(x)$ is finite for all real or complex values of x.

Cauchy gave an improved proof in 1827 in his "*Mémoire sur l'application du calcul des résidus á la solution des problémes de physique mathematique*," but that proof is also defective because of his having wrongly applied the first mean value theorem of the Integral Calculus. Poisson's attempt at a proof of Fourier's theorem was made in 1823 (*Journal de l'E'cole Polytechnique*, Cahier 19, p. 404), and is defective because it begs the question by *assuming* that

$$\lim_{r=1} \sum_1^\infty r^n \int_{-\pi}^{\pi} f(a)\cos n\,(a-x)da \text{ is equal to } \sum_1^\infty \int_{-\pi}^{\pi} f(a)\cos n(a-x)da,$$

there was no rigorous proof of Fourier's theorem. It may be also noted that up to 1864, on account of the weight of the authority of Dirichlet and Riemann,* it was an orthodox doctrine among mathematicians that the continuity of a function implied the convergence of the Fourier's series corresponding to it.

§ 2.

2. I proceed to give first six important criteria for the convergence at a point x_0, where

$$\lim_{t=0}\{f(x_0+t)+f(x_0-t)\}$$

exists, and indicate the relations between them. These are the criteria of Lipschitz, Dini, Jordan, de la Vallée Poussin, W. H. Young and Lebesgue, and may be stated as follows in terms of $\phi(t)$ where

$$\phi(t)=f(x_0+2t)+f(x_0-2t)-2S,$$

S being the sum of the series at x_0, and equal to

$$\tfrac{1}{2}\lim_{t=0}\{f(x_0+t)+f(x_0-t)\}.$$

* Although the first solid investigation of the proof of Fourier's theorem is due to Dirichlet who showed in his famous paper (*Crelle's Journal*, Bd. 4, p. 158) that a function, which (1) is continuous in its domain at every point with the exception of a finite number of points at which it may have ordinary discontinuities and (2) has a finite number of maxima and minima in its domain, is expansible in Fourier series; Dirichlet exaggerated the value of his proof by stating in the fourth chapter of the aforesaid paper that Fourier's theorem was applicable to *every* continuous function. According to an oral communication made by Weierstrass to P. Du Bois-Reymond, Dirichlet never ceased to believe that continuity implied expansibility in Fourier series. And in fact this was also the belief of Riemann as the following quotations from his writings show: "modern researches have made us see that there exist analytical expressions by which every continuous function can be represented in a given interval" (*Dissertation*, §1); "Indeed for every case that occurs in Nature, the question" of the representabilit of a function by Fourier series has been completely solved by Dirichlet, for "if we realise how the forces and the states of matter vary with the place and with the time in infinitesimals, we may admit in all security that the functions to which the researches of Dirichlet do not apply are not encountered in Nature" (*Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, §III). Du Bois-Reymond, who was the first to discover a continuous function for which the Fourier series is not everywhere convergent, says that he himself was at first fairly convinced of the truth of the doctrine and that it was only after he had made "hundred attempts, which all turned out to be wrong," to establish the doctrine, that he arrived at the conviction that Dirichlet's opinion was wrong.

I. *Lipschitz's Criterion* :* The Fourier's series converges at x_0 to S, if for all values of t not greater than some fixed positive number ϵ,

$$| \phi(t) | \quad \leq A t^k,$$

where A and k are fixed positive numbers.

II. *Dini's Criterion* :† The Fourier's series converges at x_0 to S, if ϵ can be so chosen that $\frac{| \phi(t) |}{t}$ is summable in the interval (o, ϵ).

III. *Jordan's Criterion* :‡ The Fourier's series converges at x_0 to S, if a neighbourhood of x_0 can be found such that $f(x)$ is of bounded variation in it.

IV. *de la Vallée Poussin's Criterion* :§ The Fourier's series converges at x_0 to S, if

(a) an interval (o, ϵ) can be found such that for values of h in that interval

$$\frac{1}{h}\int_0^h \phi(t)\, dt$$

has bounded variation, and

(b) $$\lim_{h=0} \frac{1}{h}\int_0^h \phi(t)dt = 0.$$

V. *W. H. Young's Criterion* :$ The Fourier's series converges at x_0 to S, if

(a) to the immediate right of $t=o$, $t\phi(t)$ is of bounded variation and

(b) $$\int_0^h | d\{t\phi(t)\} | = 0(h),$$

the integral meaning the total variation of $t\phi(t)$ in $(0, h)$.

VI. *Lebesgue's Criterion* :|| The Fourier's series converges at x_0 to S, if

* *Crelle's Journal*, Bd. 63, 1864, pp. 296-308.

† *Serie di Fourier, etc.*, Pisa, 1880, p. 102.

‡ *Comptes Rendus*, t. 92, 1881, pp. 228-230.

§ *Rendiconti del Circolo mat. d. Palermo*, t. 31, 1911, pp. 296-299.

$ *Comptes Rendus*, t. 163, 1916, pp. 187-190; and *Proc. L. M. S.*, Ser. 2, Vol. 17, 1918, p. 206.

|| *Math. Ann.*, Bd. 61, 1905, pp. 251-280.

(a) $$\lim_{\delta=0} \frac{1}{\delta}\int_0^{\delta} | \phi(t) | dt=0,$$

and

(b) $$\lim_{\delta=0} \int_{\delta}^{\epsilon} \frac{| \phi(t+\delta)-\phi(t) |}{t} dt=0,$$

ϵ being a constant greater than δ.

Lebesgue's criterion includes all the preceding criteria. Dini's criterion includes Lipschitz's but not Jordan's, and Jordan's does not include Dini's; de la Vallée Poussin's criterion does not include Young's and Young's does not include de la Vallée Poussin's; Young's criterion does not include Dini's.

3. If $$\lim_{t=0} \left\{ f(x_0+t)+f(x_0-t) \right\}$$

does not exist, and $f(x_0+t)+f(x_0-t)$ is of the form

$$\chi(t) \cos \psi(t),$$

$\chi(t)$ and $\psi(t)$ being monotone functions in the neighbourhood of $t=0$ and $\psi(t)$ tending to infinity with t tending to 0; then Du Bois-Reymond's criterion, given below, is applicable:

VII. *Du Bois-Reymond's Criterion*: * The Fourier's series does not converge at x_0, except in the case when

$$\psi(t) \asymp \log \frac{1}{t^2}$$

and

$$\chi(t) \asymp \sqrt{\psi''(t)} \, . \, t,$$

and then the sum is 0.

4. For uniformity of convergence in an interval (a, b) in which $f(x)$ is continuous, the continuity at the points a, b being on both sides, the following criteria hold:—

VIII. *Hobson's Criterion*: † $$\int_0^{\mu} \left| \frac{f(x+2z)+f(x-2z)-2f(x)}{z} \right| dz$$

* "Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformeln" (*Abhandlungen d. k. bayer Akademie der Wissenschaften*, 1876), p. 37.

† *Proc. L. M. S.*, Ser. 2, Vol. 5, 1907, pp. 275-289.

should exist for all values of x in (a, b), and should converge to zero, as μ tends to 0, uniformly for all values of x in (a, b).

IX. $\int_{\delta}^{\epsilon} | \chi(t)-\chi(t+\delta) | \, dt$ converges to zero, as δ tends to 0, uniformly for all values of x in (a, b), $\chi(t)$ standing for $\frac{f(x+t)+f(x-t)-2f(x)}{t}$.

It is easily seen that Hobson's criterion is satisfied if one of the four derivates of $f(x)$, (and therefore each of the other three) is bounded in the interval (a, b). The same criterion is satisfied if Lipschitz's criterion is satisfied for all values of x in (a, b), and for all values of t not numerically greater than some fixed positive number, **A**, k being positive numbers independent of x.

§ 3.

5. I will give you an idea of what may be called the convergence defects of the Fourier's series of a continuous function; the defect being the absence of ordinary convergence at a point or points, or of uniform convergence in a sub-interval of the interval in which the series is convergent or the whole of that interval. Four questions arise in connection with the absence of ordinary convergence. They are:

(A_1) Can a continuous function exist the Fourier's series of which is divergent only at a point?

(A_2) Can a continuous function exist the Fourier's series of which is divergent only at the points of an enumerable but everywhere dense set?

(A_3) Can a continuous function exist the Fourier's series of which is divergent at the points of an everywhere dense set having the potency of the continuum?

(A_4) Can a continuous function exist the Fourier's series of which is divergent at every point?

The last question is still an open one. But I proceed to formulate answers to the other questions.

6. (A_1) In a paper* published in 1873, Du Bois-Reymond gave the first example of a continuous function the Fourier's series of which

* *Gött. Nachrichten*, 1873, pp. 571-584.

diverges at a point. This question is treated at greater length in his famous memoir.* It is shown there that, if $\rho(x)$ denote a monotone function tending to 0 with x, $\psi(x)$ another monotone function tending to infinity with x tending to 0 and $\Psi(x)$ a function which tends to infinity with $\frac{1}{x}$ *without* being monotone, then "all functions $\rho(x)\cos\psi(x)$ are, with the above definitions, representable for $x=0$, whereas by the introduction of the function $\rho(x)\cos\Psi(x)$ the region of non-representable functions is partly entered upon." The example given by Du Bois-Reymond is naturally complicated and need not be reproduced here. But Du Bois-Reymond was aware of the possibility of constructing simpler examples. For, he says: "I do not doubt that quite ordinary functions composed of trigonometrical and exponential functions are non-representable for isolated values but I began to seek for such examples at a time when I was too tired of the subject to have sufficient perseverance for further investigation." Such a simple example was given by Schwarz † in 1880 and recently simpler examples were given by Lebesgue ‡ and Fejér. §

The examples of Fejér may be considered to be the simplest and they are the following:—

(*a*) Let $\qquad \psi(m,t)=\sin(2m+1)\frac{t}{2}, \qquad (0\leq t\leq 2\pi),$

then the cosine series for the function

$$\Psi(t)=\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\,\psi(2^{\nu^3},t)$$

diverges at $t=0$.

(*b*) Let $\qquad \phi(m,t)=\sum_{n=1}^{m}\frac{1}{n}\cos(m-n+1)t-\sum_{n=1}^{m}\frac{1}{n}\cos(m+n)t,$

then the cosine series for the function

$$\Phi(t)=\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\,\phi(2^{\nu^3},t)$$

* l.c.

† *Zeitschrift für Mathematik und Physik*, Bd. 25, 1880, pp. 271 f.

‡ *Lecons sur les séries trigonometriques*, Paris, 1906.

§ For (*a*) see *Crelle's Journal*, Bd. 137, 1910; for (*b*) see *Rendiconti*, t. 28, 1909; for (*c*) see *Crelle's Journal*, Bd. 138, 1910.

diverges at $t=0$.

(*c*) Generally, let

$$\phi(M, m, t) = \sum_{n=1}^{m} \frac{1}{n} \cos(M+m-n+1)t - \sum_{n=1}^{m} \frac{1}{n} \cos(M+m+n)t,$$

then the cosine series for the function

$$\Phi(t) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \phi(M_\nu, m_\nu, t)$$

diverges at $t=0$, m_ν standing for 2^{ν^3}, M_1 being 0 and M_ν standing for $2(m_1+m_2+\ldots+m_{\nu-1})$.

7. (A_2) This question had been answered by Steinhaus * in the affirmative as early as 1912, but the first example in proof of his assertion was published by him in 1919 and by Neder in 1920.†

(A_3) The answer to this question was known to be an affirmative by Steinhaus as early as 1912, but it was in 1914 that he published the theorem, ‡ that in the case of every Fourier's series whose partial sums are unbounded in an everywhere dense set of the interval $(0, 2\pi)$ the same holds true in a set which in every sub-interval of $(0, 2\pi)$ is of the potency of the continuum. Thus it follows from this theorem that Fejér's example, || *viz.*, $\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \phi(M_\nu, m_\nu, \nu!\, t)$, of a function whose cosine series diverges at least at the points of an enumerable and everywhere dense set is also an example of a function whose Fourier's series diverges at the points of a set which has the potency of the continuum in every subinterval of $(0, 2\pi)$.

8. With reference to another kind of convergence defect, *viz.*, the absence of uniform convergence in the case of a Fourier's series which converges at every point without exception, two questions arise. They are the following, both due to Lebesgue :— §

(B_1) Can a continuous function exist the Fourier's series of which for $(0 \leq x < 2\pi)$ is convergent at every point without exception but not uniformly convergent in any sub-interval including 0 ?

* See Neder's *Dissertation*, Göttingen, 1919, p. 26.

† *Krak. Anz.*, 1919, pp. 123-141 and *Math. Zeit.*, 1920, pp. 262-269.

‡ See the second foot-note on p. 26 of Neder's *Dissertation*.

|| *Münch. Berichte*, 1910.

§ *Comptes Rendus*, t. 141, 1905 ; *Lecons*, p. 89.

(B_3) Can a continuous function exist the Fourier's series of which for $(0 \leqq x < 2\pi)$ is convergent at every point without exception but not uniformly convergent in any sub-interval of $(0, 2\pi)$? The first question was answered in the affirmative by Lebesgue * who gave the first example of a function to illustrate his answer. The second question was answered in the affirmative by Steinhaus † who constructed the first example in support of the answer.

§ 4.

9. One of the most important contributions to the convergence problem was made by L. Fejér, who, in two communications ‡ to the Academy of Sciences of Paris, in December, 1900 and in April, 1902, established the result, that the continuity of a function at a point suffices for its summability (C1) at that point; if the function has a discontinuity of the first kind at the point x_0, the series is summable (C1) to $\frac{1}{2}\{f(x_0+0)+f(x_0-0)\}$.

By Fejér's theorem the case of a function having continuity, or discontinuity of the first kind, having been disposed of, just as the case of a function with a discontinuity of the second kind was practically disposed of by Du Bois-Reymond's researches on the limit of Dirichlet's integral for increasing n, the question of finding criteria for functions with discontinuities of the second kind became an important one. This question was taken up by Lebesgue who proved the following theorem:§

If S is a quantity such that

$$\int_0^z | f(x_0+2t)+f(x_0-2t)-2S | \, dt$$

has a differential coefficient at $z=0$ *equal* to 0, then S is the limit to which Cesáro's mean $S_n(x_0)$ tends as n tends to infinity. The validity of the above theorem is *obvious* when

$$\operatorname*{Lim}_{t=0} \left\{ f(x_0+2t)+f(x_0-2t) \right\}$$

* *C. R.* l.c. and *Leçons*, p. 88.

† *Bull. Intern. de l' Ac. d. Sc. de Cracovie*, 1912, p. 435.

‡ Later given in a complete form in *Math. Ann.*, Bd. 58, 1903.

§ See Lebesgue's paper in *Math. Ann.*, Bd. 61, 1905, p. 274. The form in which th theorem is given here is taken from Schlesinger and Plessner's book, "*Das Lebesguesche Integrale und Fouriersche Reihen*," 1926, p. 211.

exists. Therefore Lebesgue would be considered to have justification for giving his theorem, only if it were found applicable to functions for which the above limit did not exist. Very recently, G. Prasad * has examined the question of the applicability of Lebesgue's criterion to such discontinuous functions for which

$$f(x_0+2t)+f(x_0-2t)=\chi(t)\cos\psi(t),$$

χ and ψ being monotone in the neighbourhood of $t=0$ and ψ being unbounded. He has proved that for such functions

$$\int_0^z | f(x_0+2t)+f(x_0-2t) | \, dt$$

has always a differential coefficient at $z=0$ *different* from 0, whenever the differential coefficient exists. As, according to Prasad, S is in such cases of the existence of the differential coefficient always 0, the failure of Lebesgue's criterion is clear. According to Prasad,† the series is summable (C1), provided that (*a*) χ is finite and $\psi \prec \log \frac{1}{t^2}$ or (*b*) $\chi \prec \frac{1}{t}$, $\frac{\chi}{\psi'} \prec t$; the sum in each case is 0.

10. Lebesgue gave another theorem,‡ *viz.*, that the series for $f(x)$ is summable (C2) to S at every point x_0 at which

$$\int_0^z \{f(x_0+2t)+f(x_0-2t)-2S\}dt$$

has, for $z=0$, a differential coefficient of value 0. The validity of this theorem has not been questioned. But Hahn,§ has shown by an example that the existence and vanishing of the aforesaid differential coefficient is *not* sufficient for summability (C1). Prasad || has shown

* See G. Prasad's paper, "On the failure of Lebesgue's criterion for the summability (C1) of the Fourier series of a function at a point where the function has a discontinuity of the second kind" (*Bulletin of the Calcutta Mathematical Society*, Vol. 19, 1928, p. 1).

† *Bulletin of the Calcutta Mathematical Society*, Vol. 18, pp. 151-158; p. 184.

‡ l.c., p. 279.

§ *Jahresbericht d. deutschen Math.—Ver.*, Bd. 25, 1916, pp. 359-366.

|| *Bulletin of the Calcutta Mathematical Society*, Vol. 19, 1928, p. 25.

that there may be summability (C2), even if the differential coefficient is *non-existent.*

11. The interesting problem, as to how far the results of the classical theory of the convergence of Fourier's series, based directly on the investigation of Dirichlet's integral, can be deduced from the results of Fejér and Lebesgue, has received attention from Hardy * and Pollard.† By applying Hardy's theorem, *viz.*, that, if Σu_n is summable (C1) and $u_n = O(\frac{1}{n})$, Σu_n is convergent, Fejér's theorem leads to the result of Jordan, *viz.*, that the Fourier's series converges within any interval for which the variation of $f(x)$ is bounded. Using a modification of the analysis, leading to the above theorem of Hardy, Pollard deduces Dini's criterion from Fejér's theorem.

12. A number of theorems relating to convergence (Ck) may be mentioned here:—

(*a*) M. Riesz, Chapman and W. H. Young showed that Fourier's series is summable (Ck), where $k>0$, at any point of continuity or of ordinary discontinuity of the function.‡

(*b*) Hardy and Littlewood proved the following theorem: If $f(x)$ is integrable and, in addition to that, bounded in the neighbourhood of x_0, then its Fourier's series, if summable by Cesáro's mean of any one order, is also summable by Cesáro's mean of every positive order; the necessary and sufficient condition for the summability is the existence of the limit

$$\lim_{h=0} \frac{1}{2h} \int_0^h \{f(x+t)+f(x-t)\}\,dt$$

and the sum is that limit.§

(*c*) There is no condition known which is both sufficient and necessary for the summability (Ck) of the Fourier's series for a given function $f(x)$ and a given value of k. But Hardy and Littlewood have proved the following theorem:‖ The necessary and sufficient condition that the Fourier's series corresponding to $f(x)$ should be summable

* *Proc. L. M. S.*, Ser. 2, Vol. 8.

† *Proc. L. M. S.*, Ser. 2, Vol. 15, pp. 336-339.

‡ Riesz, *C. R.*, t. 149, 1909, p. 909; Chapman, *Proc. L. M. S.*, Ser. 2, Vol. 9, 1911, p. 390; Young, *Leipziger Berichte*, Bd. 63, 1911, p. 377.

§ *Proc. L. M. S.*, Ser. 2, Vol. 17, 1918, pp. XIII-XV.

‖ *Math. Zeit.*, Bd. 19, 1923, pp. 67-96.

(Ck) to S, for *some value or another* of k, at the point x_0, is that there should be an integer r such that, if

$$\phi(t)=f(x_0+2t)+f(x_0-2t)-2S,$$

$$\lim_{t=0} \phi_r(t)=0,$$

where

$$\phi_1(t)=\frac{1}{t}\int_0^t \phi(t_1)dt_1, \quad \phi_2(t)=\frac{1}{t}\int_0^t \phi_1(t_1)dt_1, \quad \phi_3(t)=\frac{1}{t}\int_0^t \phi_2(t_2)dt_2,$$

... ...

(*d*) W. H. Young * has proved that the Fourier's series of a function of bounded variation converges everywhere (C, $k-1$), when k is any positive quantity whatever.

§ 5.

13. In a paper published in 1913, Hardy and Littlewood † introduced a new notion, *viz.*, that of "strong" summability (C1) of a Fourier's series. According to them, a series is strongly summable (C1) at any point of continuity x_0 if

$$\sum_0^n |\, s_m-f(x)\,| = o(n),$$

s_m denoting as usual the sum of the first $(m+1)$ terms of the series $\sum_0^\infty (a_n \cos nx+b_n \sin nx)$. This notion is an extension of the notion of summability (C1) which gives, according to Fejér's theorem,

$$\sum_0^n \{s_m-f(x)\}=o(n).$$

In the same paper it was proved that, if $f(x)$ is of integrable square in a neighbourhood of x_0, and

$$\int_0^t \{\phi(u)\}^2 du=o(t),$$

* *Proc. L. M. S.*, Ser. 2, Vol. 17, 1918, p. 210.

† *C. R.*, t. 156, pp. 1307-1309.

then

$$\sum_0^n (s_n - S)^2 = o(n),$$

and *a fortiori*

$$\sum_0^n | s_n - S | = o(n).$$

Hardy and Littlewood say of these theorems: "The classical theorem of Fejér and its generalization by Lebesgue show that in these circumstances

$$\sum_0^n (s_n - S) = o(n).$$

The interest of our theorem is that it shows (for example) that, when the Fourier's series of a continuous function is not convergent, its summability is not merely a consequence of the cancelling of the various deviations summed up in Fejér's mean, but rather of the comparative smallness of the deviations."

14. The original results of Hardy and Littlewood were extended in certain respects by Carleman,* Sutton,† and Hardy and Littlewood themselves;‡ these extensions were criticized by G. Prasad in a paper which he read before the Calcutta Mathematical Society and which will appear in the *Bulletin* of that Society.

(*a*) Carleman proved that, if

$$\int_0^t \{\phi(u)\}^2 du = O(t),$$

and

$$\int_0^t | \phi(u) | du = o(t),$$

then

$$\sum_0^n | s_n - S |^q = o(n)$$

for all positive values of q.

* *Proc. L. M. S.*, Ser. 2, Vol. 21, 1923, pp. 483-492.
† *Proc. L. M. S.*, Ser. 2, Vol. 23, 1925, pp. XLVIII-LI.
‡ *Proc. L. M. S.*, Ser. 2, Vol. 26, 1927, pp. 273-286.

(*b*) Sutton proved (*i*) that Carleman's result holds even if the second condition in (*a*) is replaced by the less restrictive condition

$$\int_0^t |\phi(u)|^p du = O(t),$$

where p is any number greater than 1; and (*ii*) that if the second condition is replaced by

$$\int_0^t \phi(u)du = o(t),$$

then

$$\sum_0^n |\sigma_m - S|^q = o(n),$$

for every positive q, σ_m being the mth Fejér mean.

(*c*) The latest theorem of Hardy and Littlewood attempts to generalize Carleman's theorem still further and proves, that if $p > 1$ and

$$\int_0^t |\phi(u)|^p du = O(t), \qquad \int_0^t \phi(u)du = o(t),$$

then

$$\sum_0^n |s_m - S|^q = o(n),$$

for every positive q.

(*d*) G. Prasad's criticism is on the same lines as his criticism of Lebesgue's criterion for summability (C1) and is directed to prove the failure of the above theorem of Hardy and Littlewood for the case in which

$$f(x_0+2t)+f(x_0-2t) = \chi(t)\cos\psi(t),$$

χ and ψ being both unbounded and monotone in the neighbourhood of $t=0$.

§ 6.

15. Before I conclude this historical introduction, I should like to mention briefly a number of interesting results relating (*a*) to the properties of the Fourier coefficients of a function, (*b*) to the question of the uniqueness of the representation of a function by Fourier's series or Cesáro's mean of a given order, and (*c*) to the analogy between the properties of the Fourier's series of a function and the Legendre or Bessel series of the same function.

(*a*) Riemann's well-known result that, with increasing n, a_n and b_n converge to 0 if $f(x)$ is absolutely integrable, has been generalized by many writers and some of the results are

(*i*) If $f(x)$ is of summable square,

$$\sum_1^\infty (a_n^2+b_n^2)$$

is convergent.

(*ii*) If $f(x)$ is of bounded variation, na_n and nb_n are bounded, but do not necessarily tend to 0.*

(*iii*) If

$$|f(x+\delta)-f(x)| < C\delta,$$

then

$$\sum_1^\infty n^2(a_n^2+b_n^2)$$

is convergent.†

(*b*) If the trigonometrical series $\frac{1}{2}a_0+\sum_{n=0}^{\infty}(a_n\cos nx+b_n\sin nx)$ converges almost everywhere to the values of a function $f(x)$ which is summable in $(-\pi, \pi)$, and the set of points at which it does not converge, or oscillate between finite limits, contains no perfect component, then the series is a Fourier's series, and it is the unique trigonometrical series which satisfies the prescribed conditions.‡

The question of the existence of a similar theorem about convergence (C1) has not been completely settled as yet.

* F. Riesz., *Math. Zeit.*, Bd. 2, 1918, pp. 312-315.

† Fatou, *Acta Math.*, Bd. 30, 1906.

‡ Hobson, *Theory of Functions of a Real Variable*, Vol. 2, p. 677. See also Lebesgue's *Lecons*, p. 122.

(*c*) So far as ordinary convergence is concerned, there is a close analogy * between the Fourier series of a function and the Legendre or Bessel series of that function. But, as shown by Fejér and Prasad, for convergence (C1) the Legendre series does not always behave as the Fourier series. For example, Fejér † has shown that, in case of divergence, whereas for the Fourier series the Cesáro mean of a bounded function lies between the upper and lower limits of the function with increasing n, for the Legendre series in the same circumstances this is not the case.

* See W. H. Young's papers in *Proc. L. M. S.*, Ser. 2, Vol. 18.

† *Math. Ann.*, Bd. 67.

SECOND LECTURE.

Various Criteria for Convergence at a Point or in an Interval.

§ 7.

16. To-day's lecture will deal at some length with the various criteria for the convergence of a Fourier's series at a point or its uniform convergence in an interval ; I will give the proofs of many of those criteria and will also establish the statements made in the first lecture about the mutual relations between the various criteria. As, throughout this lecture, I shall concern myself with functions, integrable in the sense of Riemann, unless the contrary is explicitly stated, I will begin by giving the proof of the following theorem first enunciated by Riemann :—*

If $\phi(x)$ be an integrable function in an interval (α, β), then

$$\lim_{\mu=\infty} \int_{\alpha}^{\beta} \phi(x) \begin{matrix}\cos \\ \sin\end{matrix} \mu x \, dx = 0. \qquad (T_1)$$

Proof :

Consider the integral

$$I_\mu = \int_{\alpha}^{\beta} \phi(x) \sin \mu(x-a) dx.$$

Starting from a, divide the interval (α, β) into a number of sub-intervals of the form

$$a + \frac{s\pi}{\mu} \, , \; a + \frac{(s+2)\pi}{\mu} \, .$$

Then

I_μ = the sum of integrals $\int_{a+\frac{s\pi}{\mu}}^{a+\frac{(s+2)\pi}{\mu}} \phi(x) \sin \mu(x-a) dx$ which may be denoted by $I_{\mu,s}$.

* *Abhandlungen d. k. Gesellschaft d. Wiss. zu Göttingen*, Bd. XIII ; also see Riemann's *Werke*, Bd. 1.

Now in $I_{\mu,s}$ the sine is positive in one half of the interval and negative in the other half. If, therefore, we denote by M_s and m_s, respectively, the upper and lower limits of $\phi(x)$ in the interval, the value of $I_{\mu,s}$ is *increased* by substituting, for $\phi(x)$, M_s in the former half and m_s in the latter half; similarly the value of $I_{\mu,s}$ is *diminished* by substituting, for $\phi(v)$, m_s in the former half and M_s in the latter half. In the first case, the value obtained is

$$\frac{2}{\mu}(M_s - m_s)$$

and in the second case the value obtained is

$$\frac{2}{\mu}(m_s - M_s),$$

Therefore

$$\left| I_{\mu,s} \right| < \frac{2}{\mu}(M_s - m_s),$$

and, consequently,

$$\left| I_\mu \right| \text{ being less than } \Sigma \left| I_{\mu,s} \right|$$

is less than $\frac{2}{\mu}\Sigma(M_s - m_s)$.

But, because of the integrability of $\phi(x)$, the quantity $\frac{2}{\mu}\Sigma(M_s - m_s)$ tends to 0 as the interval $\frac{2\pi}{\mu}$ is made smaller and smaller. Therefore I_μ tends to 0 with $\frac{1}{\mu}$.

Thus the theorem is proved, as, putting a equal to 0, we get

$$\int_a^\beta \phi(x) \sin \mu x \, dx,$$

and, putting a equal to $-\frac{\pi}{2\mu}$, we get

$$\int_a^\beta \phi(x) \cos \mu x \, dx.$$

§ 8.

17. I proceed to deduce from Riemann's theorem the following theorem which gives the *necessary and sufficient condition* for the convergence of the Fourier's series corresponding to $f(x)$ in the interval $(-\pi, \pi)$:—

The condition which is necessary and sufficient for the convergence of the series to S at x is that

$$\lim_{\mu=\infty} \frac{1}{\pi}\int_0^{\epsilon} \phi(t).\ \frac{\sin \mu\, \pi t}{t} dt = 0,$$

where ϵ is a small positive quantity and

$$\phi(t) = f(x+2t) + f(x-2t) - 2S. \qquad (T_2)$$

Proof :

Denoting by s_n the sum of the first $(2n+1)$ terms of the series, we have

$$s_n \equiv \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \{\frac{1}{2} + \sum_{m=1}^{n} \cos m(t-x)\} f(t) dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin \frac{2n+1}{2}(t-x)}{2 \sin \frac{t-x}{2}} f(t) dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin (2n+1)\frac{t}{2}}{2 \sin \frac{t}{2}} f(x+t)\, dt, \qquad (1)$$

remembering that $f(t+2\pi) = f(t)$.

Now, let ϵ_1 be an arbitrarily small quantity >0 but fixed after choice ; then we can put

$$s_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\epsilon_1} + \int_{-\epsilon_1}^{\epsilon_1} + \int_{\epsilon_1}^{\pi}\right\};$$

and Riemann's theorem is applicable to the first and third parts of s_n,

as $$f(x+t)\frac{\sin\frac{2n+1}{2}t}{\sin\frac{t}{2}}=\frac{f(x+t)\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin nt+f(x+t).\cos nt$$

and as $$\frac{f(x+t)\cos\frac{t}{2}}{\sin\frac{t}{2}}$$ is integrable in the intervals $(-\pi, -\epsilon_1)$ and (ϵ_1, π). Therefore, with increasing n, s_n behaves as

$$\frac{1}{\pi}\int_{-\epsilon_1}^{\epsilon_1} f(x+t)\ \frac{\sin\frac{2n+1}{2}t}{2\sin\frac{t}{2}}\ dt$$

i.e., as $$\frac{1}{\pi}\int_{-\epsilon}^{\epsilon} f(x+2t)\ \frac{\sin(2n+1)t}{t}\ dt,$$

i.e., as $$\frac{1}{\pi}\int_{0}^{\epsilon} \{f(x+2t)+f(x-2t)\}\ \frac{\sin(2n+1)t}{t}\ dt,$$

where $\epsilon=2\epsilon_1$.

Now it is known that

$$\frac{2}{\pi}\int_{0}^{\epsilon} \frac{\sin(2n+1)t}{t}\ dt$$

tends to 1 with increasing n. Therefore $\underset{n=\infty}{\text{Lim}}\ s_n$ will be equal to S, then and only then, when

$$\underset{n=\infty}{\text{Lim}}\int_{0}^{\epsilon} \{f(x+2t)+f(x-2t)-2S\}\frac{\sin(2n+1)t}{t}\,dt=0,$$

i.e., $$\underset{n=\infty}{\text{Lim}}\int_{0}^{\epsilon} \phi(t)\ \frac{\sin(2n+1)t}{t}\ dt=0.$$

§ 9.

18. I proceed first to deduce from (T_2) and (T_1) the six important criteria which I gave in my first lecture; they are the criteria I-VI.

I. *Lipschitz's Criterion*: If

$$| \phi(t) | \leq At^k, \quad k > 0,$$

then $\frac{\phi(t)}{t}$ is integrable in $(0, \epsilon)$ and, consequently, from (T_1) it follows that

$$\operatorname{Lim}_{n=\infty} \int_0^{\epsilon} \frac{\phi(t)}{t} \cdot \sin (2n+1)t \, dt = 0.$$

Therefore the convergence of the series follows from (T_2).

II. *Dini's Criterion*: If $\frac{| \phi(t) |}{t}$ is integrable in $(0, \epsilon)$, the above reasoning is applicable and convergence is proved.

III. *Jordan's Criterion*: If a neighbourhood of x_0 can be found such that $f(x)$ is of bounded variation in that neighbourhood, then also for $\phi(t)$ such a neighbourhood about $t=0$ exists; let this be $(0, \delta)$.

Now ϕ can be expressed as the difference

$$\phi_1 - \phi_2$$

of two functions ϕ_1 and ϕ_2, each monotone and increasing. Let

$$S_1 = \operatorname{Lim}_{t=0} \phi_1(t),$$

$$S_2 = \operatorname{Lim}_{t=0} \phi_2(t),$$

$$J_n = \int_0^{\delta} (\phi_1 - S_1) \frac{\sin (2n+1)t}{t} \, dt,$$

$$K_n = \int_0^{\delta} (\phi_2 - S_2) \frac{\sin (2n+1)t}{t} \, dt.$$

Then, since obviously

$$0 = \lim_{t=0} \phi(t),$$

$$S_1 = S_2.$$

Hence

$$J_n - K_n = \int_0^\delta \phi(t) \frac{\sin(2n+1)t}{t}\, dt.$$

Choose δ so small that, in the interval $(0, \delta)$, for a prescribed $g > 0$, the inequality

$$\phi_1(t) - S_1 \leq g \text{ holds.}$$

Then, by the second mean value theorem,

$$J_n = g \int_\xi^\delta \frac{\sin(2n+1)t}{t}\, dt, \quad (0 \leq \xi \leq \delta).$$

But it can be proved * that

$$\left| \int_\xi^\delta \sin \frac{(2n+1)t}{t}\, dt \right| < 2\pi, \quad \text{however large } n \text{ may be.}$$

* $$\int_\xi^\delta \frac{\sin(2n+1)t}{t} dt = \int_{(2n+1)\xi}^{(2n+1)\delta} \frac{\sin t}{t} dt = \int_0^{(2n+1)\delta} - \int_0^{(2n+1)\xi} \frac{\sin t}{t}.$$

But $\int_0^\alpha = \int_0^\pi + \int_\pi^{2\pi} + \ldots < \int_0^\pi$ since each member of the right side of the equality is alternately positive and negative.

$\therefore$ $\int_0^{(2n+1)\xi}$ is numerically less than $\int_0^\pi \frac{\sin t}{t}\, dt < \pi$.

Similarly for $\int_0^{(2n+1)\delta}$. Hence the difference

$\int_{(2n+1)\xi}^{(2n+1)\delta}$ is numerically less than 2π.

Therefore

$$| J_n | < 2\pi g.$$

But δ can be chosen to be so small that g is as small as we please. Therefore J_n tends to 0. Similarly it can be shown that K_n tends to 0. Therefore

$$\operatorname*{Lim}_{n=\infty} \int_0^{\delta} \phi(t) \frac{\sin (2n+1)t}{t}\, dt = 0,$$

and, consequently, from (T_2) the convergence of the series follows.

IV. *Vallée Poussin's Criterion* : (*a*) Let $F(u) \equiv \frac{1}{u}\int_0^u \phi(t)dt$ have bounded variation in an interval $(0, \epsilon)$, then, according to a well-known theorem

$$\operatorname*{Lim}_{\mu=\infty} \int_0^{\epsilon} F(u). \frac{\sin \pi\mu u}{u}\, du = \frac{\pi}{2} F(+0) \text{ which is } 0.$$

Now according to theorem (T_2), in order to establish the convergence of the Fourier's series at x_0 to S, we have to prove that

$$\operatorname*{Lim}_{\mu=\infty} \int_0^{\epsilon} \phi(t). \frac{\sin \mu\pi t}{t}\, dt = 0.$$

Integrating the above integral by parts, we have

$$\left[\frac{\sin \mu\pi u}{u} \cdot uF(u)\right]_0^{\epsilon} + \int_0^{\epsilon} uF(u). \frac{\sin \mu\pi u}{u^2} du - \mu\pi \int_0^{\epsilon} F(u). \cos \mu\pi u\, du$$

$$= F(\epsilon) \sin \mu\pi\epsilon - \mu\pi \int_0^{\epsilon} F(u) \cos \mu\pi u\, du \text{ (in the limit)}$$

$$= \{F(\epsilon) - F(+0)\} \sin \mu\pi\epsilon - \mu\pi \int_0^{\epsilon} \{F(u) - F(+0)\} \cos \mu\pi u\, du.$$

But $F(\epsilon) - F(+0)$ can be made as small as we please by choosing ϵ sufficiently small. We have therefore to prove only this : that

$$\mu\pi \int_0^{\epsilon} \{F(u) - F(+0)\} \cos \mu\pi u\, du$$

tends to o with increasing μ.

(*b*) Now let T (u) denote the total variation of $F(u)$ in (o, u). This function being positive and non-decreasing, we have by the second mean value theorem,

$$\int_0^{\epsilon} T(u)\ \mu\pi \cos \mu\pi u\ du = T(\epsilon) \int_{\epsilon'}^{\epsilon} \mu\pi \cos \mu\pi u\ du,$$

ϵ' being less than ϵ.

Similarly, ϵ'' being another quantity less than ϵ,

$$\int_0^{\epsilon} \{T(u) - F(u) + F(+o)\} \mu\pi \cos \mu\pi u\ du$$

$$= \{T(\epsilon) - F(\epsilon) + F(+o)\} \int_{\epsilon''}^{\epsilon} \mu\pi \cos \mu\pi u\ du$$

Now $\mu\pi \int_0^{\epsilon} \{F(u) - F(+o)\} \cos \mu\pi u\ du$ is the difference of the above two integrals which themselves are both as small as we please, with suitably chosen ϵ ; for, $| \int \mu\pi \cos \mu\pi u\ du | < 2$.

Thus Vallée Poussin's criterion is established, as

$$\operatorname*{Lim}_{h=0} \frac{1}{h} \int_0^h \phi(t)\ dt = 0$$

gives simply $S = \frac{1}{2} \{f(x_0 + 0) + f(x_0 - 0)\}$.

V. *Young's Criterion*: Young gave his criterion in the following form : If $f(x)$ is simply discontinuous at the point x and if in a neighbourhood of this point, as small as we please, we have

$$\frac{1}{2}[f(x+h) + f(x-h))] = \frac{1}{h} \int_0^h g(t)\ dt,$$

$g(t)$ being a bounded function or more generally such that

$$\frac{1}{h} \int_0^h | g(t) |\ dt$$

is a bounded function of h; then the Fourier's series of $f(x)$ converges at the point x to $\frac{1}{2}\{f(x+0) + f(x-0)\}$.

(*a*) To prove this result it suffices to take the origin for the given point and to suppose $f(x)$ to be an even function. Now if

$$s_n = a_1 + a_2 + \ldots + a_n \ , \ \mathrm{C}(n\,a_n) = \frac{a_1 + 2a_2 + \ldots + na_n}{n} \ ,$$

we have $$s_n - \mathrm{C}(na_n) = \frac{s_1 + s_2 + \ldots + s_{n-1}}{n} \ , \qquad \ldots \ (1)$$

a_n being the co-efficient of $\cos nx$ in the Fourier's series of $f(x)$.

Since $f(+0)$ exists and $f(x)$ is bounded in the neighbourhood of the origin, by Fejér's theorem

$$\lim_{n=\infty} \frac{s_1 + s_2 + \ldots + s_{n-1}}{n-1} = f(+0).$$

Hence by using (1) we have the Fourier's series converging for $x=0$ if

$$\lim_{n=\infty} \mathrm{C}(n\,a_n) = 0.$$

But

$$\frac{1}{2}\pi \mathrm{C}(na_n) = \mathrm{C}\left\{\int_0^\pi n \cos nx\, f(x)\, dx\right\}$$

$$= \frac{1}{2n}\int_0^\pi \frac{d}{dx}\left[\cot\frac{x}{2} - \mathrm{cosec}\,\frac{x}{2}\cos\left(n+\frac{1}{2}\right)x\right] f(x)dx.$$

Therefore

$$\frac{1}{2}\pi \lim_{n=\infty} \mathrm{C}(na_n) = \lim_{n=\infty}\left[\frac{1}{2n}\int_0^\pi \frac{d}{dx}\left(\frac{1}{\frac{x}{2}} - \frac{\cos nx}{\frac{x}{2}}\right) f(x)dx\right] \qquad (2)$$

(*b*) Now divide the interval (o, π) into three parts : (o, p), (p, e) and (e, π), where $p = \frac{2\mathrm{P}\pi}{n}$, P being any integer and n being sufficiently large. But our result depends only on the character of $f(x)$ in the near neighbourhood of $x=0$. We may therefore take $f(x)=0$ in the interval (e, π). In the interval (o, p), put $x = \frac{t}{n}$; $f\left(\frac{t}{n}\right)$ is bounded and has $f(+o)$ as limit with increasing n ; we may therefore multiply

it by the bounded function $\frac{d}{dx}\left(\frac{1-\cos t}{t}\right)$ and integrate term by term. We shall have then

$$\operatorname*{Lim}_{n=\infty} \frac{1}{n}\int_0^p f(x)\frac{d}{dx}\left(\frac{1-\cos nx}{x}\right)dx$$

$$= \operatorname*{Lim}_{n=\infty} \int_{\bullet}^{2P\pi} f\left(\frac{t}{n}\right)\frac{d}{dt}\left(\frac{1-\cos t}{t}\right)dt$$

$$= \int_0^{2P\pi} f(+o)\cdot\frac{d}{dt}\left(\frac{1-\cos t}{t}\right)dt = 0.$$

Hence (2) becomes

$$\frac{1}{2}\pi \operatorname*{Lim}_{n=\infty} C(na_n) = \operatorname*{Lim}_{n=\infty} \int_p^e f(x)\frac{d}{dx}\left(\frac{1-\cos nx}{nx}dx\right)$$

$$= \operatorname*{Lim}_{n=\infty} \int_p^e xf(x)\frac{d}{dx}\,q_n dx$$

$$= \operatorname*{Lim}_{n=\infty} \left\{ [xf(x)q_n(x)]_p^e - \int_p^e g(x)q_n(x) \right\}, \text{ where } |q_n|$$

$$= \left| \frac{1-\cos nx}{nx^2} - \int_x^e \frac{1-\cos nx}{nx^3}\,dx \right|$$

$$\leq \frac{2}{nx^2} + \int_x^e \frac{2}{nx^3}\,dx < \frac{3}{nx^2}.$$

It follows that $\left| \frac{\pi}{6} \operatorname*{Lim}_{n=\infty} \mathrm{C}\,(na_n) \right|$

$$< \operatorname*{Lim}_{n=\infty} \left\{ \left| \left[\frac{f(x)}{nx} \right]_p^e \right| + \int_p^e \frac{|g|}{nx^2}\, dx \right.$$

$$< \frac{f(p)}{2\mathrm{P}\pi} + \operatorname*{Lim}_{n=\infty} \left\{ \left| \left[\frac{1}{nx^2} \int_0^x |g|\, dx \right]_p^e \right| + \int_p^e \left[\frac{2}{nx^3} \int_0^x |g|\, dx \right] dx \right\}$$

$$\leq \frac{f(p)}{2\mathrm{P}\pi} + \frac{\mathrm{B}}{2\mathrm{P}\pi} + \operatorname*{Lim}_{n=\infty} 2\mathrm{B} \int_p^e \frac{dx}{nx^2} \leq \frac{f(p)+3\mathrm{B}}{2\mathrm{P}\pi},$$ B being the upper

limit of $\frac{1}{x}\int_0^x |g|\, dx$. So it follows easily that $\operatorname*{Lim}_{n=\infty} \mathrm{C}(na_n)=0$.

19. VI. *Lebesgue's Criterion* : (*a*) If I_n denote $\int_0^\delta \phi(t) \frac{\sin \frac{2n+1}{2} t}{t} dt$,

then, putting ν for $n+\frac{1}{2}$, we have

$$\mathrm{I}_n = \int_0^{\frac{2\pi}{\nu}} + \int_{\frac{2\pi}{\nu}}^{\delta + \frac{\pi}{\nu}} - \int_\delta^{\delta + \frac{\pi}{\nu}}, \qquad \ldots \quad (1)$$

the integrand in each integral being $\phi(t)\ \frac{\sin\ \nu t}{t}$.

Now

$$\int_{\frac{2\pi}{\nu}}^{\delta + \frac{\pi}{\nu}} \phi(t) \frac{\sin \nu t}{t} dt = -\int_{\frac{\pi}{\nu}}^{\delta} \phi\left(t + \frac{\pi}{\nu}\right) \frac{\sin \nu t}{t + \frac{\pi}{\nu}} dt. \qquad \ldots \quad (2)$$

Therefore, using the above in (1), and adding (1) and

$$I_n = \int_0^{\frac{\pi}{v}} + \int_{\frac{\pi}{v}}^{\delta},$$

we have

$$2\,I_n = \int_0^{\frac{\pi}{v}} + \int_0^{\frac{2\pi}{v}} + \int_{\frac{\pi}{v}}^{\delta} \left\{ \frac{\phi(t)}{t} - \frac{\phi\left(t+\frac{\pi}{v}\right)}{t+\frac{\pi}{v}} \right\} \sin vt\, dt$$

$$- \int_{\delta}^{\delta + \frac{\pi}{v}}, \qquad \ldots \qquad \ldots \quad (3)$$

the integrand in each of the 1st, 2nd and 4th integrals being $\frac{\sin vt}{t}\,\phi$.

(*b*) Denote the four integrals in (3) by $2i_1$, $2i_2$, $2i_3$, $2i_4$ respectively. Now in $2i_1$ and $2i_2$,

$$\left| \frac{\sin vt}{t} \right| \leq v,$$

therefore

$$|\, i_1 \,| + |\, i_2 \,| \leq \frac{v}{2} \int_0^{\frac{\pi}{v}} |\, \phi \,|\, dt + \frac{v}{2} \int_0^{\frac{2\pi}{v}} |\, \phi \,|\, dt$$

$$\leq v \int_0^{\frac{2\pi}{v}} |\, \phi \,|\, dt = v\psi\left(\frac{2\pi}{v}\right),$$

where $\psi(t)$ stands for $\int^t |\, \phi \,|\, dt$.

Now, by the first part of Lebesgue's criterion, $\psi'(0)=0$. Therefore

$$\lim_{\nu=\infty}\{i_1+i_2\}=0.$$

Further,

$$|i_4| \leq \frac{1}{2}\int_{\delta}^{\delta+\frac{\pi}{\nu}} \left|\frac{\phi}{t}\right| dt$$

and therefore also $\lim_{\nu=\infty} i_4=0.$

Thus for the determination of $\lim_{n=\infty} I_n$ we have only i_3 to investigate.

(c) Now

$$i_3=\frac{1}{2}\int_{\frac{\pi}{\nu}}^{\delta}\left\{\frac{\phi}{t}-\frac{\phi\left(t+\frac{\pi}{\nu}\right)}{t}+\frac{\phi\left(t+\frac{\pi}{\nu}\right)}{t}-\frac{\phi\left(t+\frac{\pi}{\nu}\right)}{t+\frac{\pi}{\nu}}\right\}\sin \nu t\ dt$$

$$=\frac{1}{2}\int_{\frac{\pi}{\nu}}^{\delta}\frac{\phi(t)-\phi\left(t+\frac{\pi}{\nu}\right)}{t}\sin \nu t\ dt+\frac{\pi}{2\nu}\int_{\frac{\pi}{\nu}}^{\delta}\frac{\phi\left(t+\frac{\pi}{\nu}\right)}{t\left(t+\frac{\pi}{\nu}\right)}\sin \nu t\, dt.$$

Call the above two parts i_5 and i_6 respectively. Then

$$|i_6| \leq \frac{\pi}{2\nu}\int_{\frac{\pi}{\nu}}^{\delta}\frac{\left|\phi\left(t+\frac{\pi}{\nu}\right)\right|}{t^2}dt.$$

But $$\int_{\frac{\pi}{\nu}}^{t}\left|\phi\left(t+\frac{\pi}{\nu}\right)\right| dt=\int_{\frac{2\pi}{\nu}}^{t+\frac{\pi}{\nu}}|\phi|\,dt \leq \psi\left(t+\frac{\pi}{\nu}\right).$$

Therefore, by means of integration by parts,

$$|i_6| \leq \frac{\pi}{2v} \frac{\psi\left(\delta+\frac{\pi}{\nu}\right)}{\delta^2} + \frac{\pi}{\nu}\int_{\frac{\pi}{v}}^{\delta} \frac{\psi\left(t+\frac{\pi}{v}\right)dt}{t^3}.$$

Now, for sufficiently great v,

$$\frac{1}{v}\frac{\psi\left(\delta+\frac{\pi}{\nu}\right)}{\delta^2} \leq \frac{1}{v}\frac{\psi(2\delta)}{\delta^2}$$

and therefore

$$\operatorname*{Lim}_{v=\infty} \frac{1}{v}\frac{\psi\left(\delta+\frac{\pi}{v}\right)}{\delta^2} = 0.$$

Again, in the interval $\left(\frac{\pi}{v}, \delta\right)$,

$$\frac{\psi\left(t+\frac{\pi}{v}\right)}{t} \leq \frac{\psi(2t)}{t}, \text{ and } \operatorname*{Lim}_{t=o} \frac{\psi(2t)}{t} = 0;$$

therefore, if $\omega(\delta)$ denote the upper limit of $\frac{\psi(2t)}{t}$ for $0<t<\delta$,

$$\left|\frac{\pi}{v}\int_{\frac{\pi}{v}}^{\delta} \frac{\psi\left(t+\frac{\pi}{v}\right)}{t^3}dt\right| \leq \omega(\delta)\cdot\frac{\pi}{v}\int_{\frac{\pi}{v}}^{\delta}\frac{dt}{t^2} < \omega(\delta),$$

and this can be made as small as we please by choosing δ to be sufficiently small. Thus it follows that by choosing δ to be sufficiently small and v to be sufficiently large $|i_6|$ can be made as small as we please.

(*d*) We have therefore left only i_5 to consider. This can be made as small, as we please, by suitable choice of δ and v, provided that, for a

prescribed η, it be possible to choose δ so small and N so large that, for every $v>N$,

$$\int_{\frac{\pi}{v}}^{\delta} \left| \frac{\phi(t)-\phi\left(t+\frac{\pi}{v}\right)}{t} \right| dt < \eta.$$

Putting ϵ for $\frac{\pi}{v}$, the above condition becomes the following : For every $\eta>0$, it should be possible to choose positive quantities δ and ϵ_1 so that for $\epsilon<\epsilon_1$,

$$\int_{\epsilon}^{\delta} \left| \frac{\phi(t)-\phi(t+\epsilon)}{t} \right| dt \quad < \eta.$$

§ 10.

20. The logical relations between the six criteria given above have been studied by a number of writers ;* the following exposition is taken from a paper by Hardy,† it being *assumed* that $S=x_0=0$ and that $f(x)$ is even.

(α) *Neither Dini's criterion nor Jordan's includes the other.* For, $f(x)=\left(\log\frac{1}{x}\right)^{-1}$ satisfies Jordan's criterion but not Dini's. And $f(x)=x^p\sin\frac{1}{x}$, $(0<p\leq 1)$, satisfies Dini's criterion but not Jordan's.

(β) *Vallée Poussin's criterion includes both Dini's and Jordan's.*

Proof :

If a function is monotone its mean value is also monotone. Hence, if $f(x)$ is of bounded variation, so also is $F(x)\equiv\frac{1}{x}\int_0^x f(t)dt$. Thus Vallée Poussin's criterion includes Jordan's.

* See S. Pollard's paper, "On the Criteria for the Convergence of a Fourier's Series" (*Journal L. M. S.*, Vol. 2, p. 255, 1927).

† *Messenger of Mathematics*, Vol. 49, 1919, pp. 149-155.

Again suppose that Dini's criterion is satisfied. We have

$$|F'| = \left| \frac{f}{x} - \frac{1}{x^2}\int_0^x f\,dt \right|$$

$$\leq \frac{|f|}{x} + \frac{1}{x^2}\int_0^x |f|\,dt \qquad \ldots \quad (1)$$

for $x>0$, at any rate with the exception of a set of values of measure 0. The first term of the right side in (1) is integrable in $(0, \delta)$. Also

$$\int_\epsilon^\delta \frac{dx}{x^2}\int_0^x |f|\,dt = \frac{1}{\epsilon}\int_0^\epsilon |f|\,dt - \frac{1}{\delta}\int_0^\delta |f|\,dt + \int_\epsilon^\delta \frac{|f|}{x}\,dx$$

and therefore it tends to a limit as ϵ tends to 0. Hence the second term of the right side in (1) is also integrable in $(0, \delta)$. Therefore $|F'|$ is integrable in $(0, \delta)$ and consequently F is of bounded variation.

(γ) *Young's criterion includes Jordan's.* For,

$$\int_0^x |d(tf)| \leq \int_0^x |f|\,dt + \int_0^x t\,|df|$$

$$= o(x) + 0\left(x\int_0^x |df|\right) = 0(x)$$

if Jordan's criterion is satisfied.

(δ) *Young's criterion does not include Dini's.* For, consider

$$f(x) = x^p \sin\frac{1}{x}, \quad (0<p\leq 1).$$

$$d(xf) = \left\{(p+1)x^p \sin\frac{1}{x} - x^{p-1}\cos\frac{1}{x}\right\}dx.$$

Now Young's criterion requires that

$$\int_0^x |t|^{p-1}\left|\cos\frac{1}{t}\right| dt = 0(x),$$

But this is only true if $p \geq 1$ whereas Dini's criterion is satisfied even when $p > 0$ and not ≥ 1.

(ϵ) *Vallée Poussin's criterion does not include Young's.* For, consider

$$f(x) = \frac{\sin lx}{lx},$$

where lx stands for $\log \frac{1}{x}$. Now Young's criterion requires that

$$\int_0^x \left| \frac{\sin lt}{lt} - \frac{\cos lt}{lt} + \frac{\sin lt}{(lt)^2} \right| dt = O(x),$$

which is obviously true. On the other hand, it is easily verified that

$$F(x) = \frac{1}{x} \int_0^x \frac{\sin lt}{lt}\, dt = \frac{\cos lx + \sin lx}{2lx} + O\left\{\frac{1}{(lx)^2}\right\},$$

$$F' = \frac{f - F}{x} = \frac{\sin lx - \cos lx}{2x\, lx} + O\left\{\frac{1}{x(lx)^2}\right\};$$

so that

$$\int_0^\delta |F'|\, dx$$

is divergent. Thus Vallée Poussin's criterion is not satisfied.

(ζ) *Lebesgue's criterion includes Vallée Poussin's and all the other criteria.*

Proof.

(a) Lebesgue's criterion may be also stated in the form

$$\int_x^\delta |\psi(t+x) - \psi(t)|\, dt = o(1), \qquad \ldots \quad (1)$$

where $\psi=\frac{f}{x}$. To see this, we observe that the difference of the integrals which occur in the above and in Lebesgue's form

$$\int_x^\delta \frac{|f(t+x)-f(t)|}{t}dt=o(1)$$

does not exceed

$$\int_\eta^\delta \frac{|f(t+x)-f(t)|}{t}dt+\int_\eta^\delta |\psi(t+x)-\psi(t)|\,dt+\int_x^\eta \frac{|f(t+x)|}{t(t+x)}dt,$$

where $x<\eta<\delta$. We can choose η so that the last integral is less than

$$\epsilon\int_x^\eta \frac{dt}{t(t+x)}<\epsilon\log 2.$$

When η is fixed, the first two integrals tend to zero, by a fundamental theorem in the theory of Lebesgue. The proposition is thus established.

(*b*) Suppose now that Vallée Poussin's criterion is satisfied. Then $f=\mathrm{F}+x\mathrm{F}'$ for almost all values of x, and (1) will certainly be satisfied if

$$\int_x^\delta |\mathrm{F}'(t+x)-\mathrm{F}'(t)|\,dt=o(1) \quad \ldots (2)$$

and

$$\int_x^\delta \left|\frac{\mathrm{F}(t+x)}{t+x}-\frac{\mathrm{F}(t)}{t}\right| dt=o(1). \quad \ldots (3)$$

Of these equations (2) is satisfied in virtue of the integrability of $\mathrm{F}'(x)$, and (3) may (by the same argument as was used above) be replaced by

$$\int_x^\delta \frac{|\mathrm{F}(t+x)-\mathrm{F}(t)|}{t}dt=o(1), \quad \ldots (4)$$

Let us write

$$\int_x^\delta \frac{|\,F(t+x)-F(t)\,|}{t}\,dt=\int_x^\eta+\int_\eta^\delta=J_1+J_2.$$

Then

$$J_1=\int_x^\eta \frac{dt}{t}\;\Big|\int_t^{t+x} F'(u\;du\;|\;\leq\;\int_x^\eta \frac{dt}{t}\int_t^{t+x}|\,F'(u)\,du\,|$$

$$=\Big[\;\log t\int_t^{t+x}|\,F'(u)\,|\,du\Big]_x^\eta-\int_x^\eta \log t\,\{\,|\,F'(t+x)\,|\,-\,|\,F'(t)\,|\,\}dt$$

$$=\log\eta\int_\eta^{\eta+x}|\,F'\,|\,du-\log x\int_x^{2x}|\,F'\,|\,du$$

$$-\int_{2x}^{\eta+x}\log\,(u-x)\,|\,F'\,|\,du+\int_x^\eta \log u\,|\,F'\,|\,du$$

$$=\int_x^{2x}\log\,\frac{u}{x}\,|\,F'\,|\,du+\int_{2x}^\eta \log\,\frac{u}{u-x}\,|\,F'\,|\,du$$

$$+\int_\eta^{\eta+x}\log\,\frac{\eta}{u-x}\,|\,F'\,|\,du.$$

In each of these integrals the logarithmic factor is bounded. Hence J_1 is less than a constant multiple of

$$\int_x^{\eta+x}|\,F'\,|\,du,$$

and may therefore be made less than ϵ by choice of η. Finally, when η is fixed, J_2 tends to 0 with x. Thus (4), and therefore (1), is satisfied.

(c) Lebesgue's criterion also includes Young's.

For, suppose Young's criterion is satisfied and write $xf=G$; also denote by $V(G)$ the variation of G in $(0, x)$. Then $G=O(x)$, $V(G)=O(x)$; and we can write

$$G=G_1-G_2$$

where G_1 and G_2 are steadily increasing functions of the form $O(x)$. And

$$\int_x^\delta \frac{|f(t+x)-f(t)|}{t}dt \leq \int_x^{mx} \frac{|f(t+x)-f(t)|}{t}dt$$

$$+\int_{mx}^\delta \left| \frac{G_1(t+x)}{t+x} - \frac{G_1(t)}{t} \right| \frac{dt}{t} + \int_{mx}^\delta \left| \frac{G_2(t+x)}{t+x} - \frac{G_2(t)}{t} \right| \frac{dt}{t}$$

$$=J+J_1+J_2, \text{ say.}$$

Now it is easily proved that $J \leq 2\mu \log m$, where μ is the upper limit of $|f|$ in the interval $0 \leq t \leq (m+1)x$. Again

$$|J_1| \leq \int_{mx}^\delta \frac{|G_1(t+x)-G_1(t)|}{t(t+x)}\, dt + x\int_{mx}^\delta \frac{|G_1(t)|}{t^2(t+x)}\, dt \ldots. \quad (5)$$

Now it is easily seen that the second term $=O(\epsilon_m)$, where ϵ_m is a function of m alone which tends to 0 with $\frac{1}{m}$. Since G_1 is monotone, the first term is

$$\int_{mx}^\delta \frac{G_1(t+x)-G_1(t)}{t(t+x)}\, dt = \int_{(m+1)x}^{\delta+x} \frac{G_1(u)}{u(u-x)}du - \int_{mx}^\delta \frac{G_1(u)du}{u(u+x)}.$$

In the first of these integrals, we may replace $u-x$ by $u+x$, the error introduced being $O(\epsilon_m)$ which may be absorbed in the second term of (5). When this is done we have

$$\left(\int_{(m+1)x}^{\delta+x} - \int_{mx}^\delta\right)\frac{G_1(u)du}{u(u+x)} = \left(\int_\delta^{\delta+x} - \int_{mx}^{(m+1)x}\right)\frac{G_1(u)}{u(u+x)}\, du \text{ which}$$

$$\text{is } O\left(\int_{\delta}^{\delta+x}\frac{du}{u+x}\right)+O\left(\int_{mx}^{(m+1)x}\frac{du}{u+x}\right)=O(\epsilon_x)+O(\epsilon_m),$$

where ϵ_x is a positive function of x alone which tends to o with x. Thus $J_1 \leq O(\epsilon_x)+O(\epsilon_m)$. Similarly for J_2. Therefore it is proved that

$$\int_x^{\delta}\frac{|f(t+x)-f(t)|}{t}\,dt < 2\mu \log m+O(\epsilon_x)+O(\epsilon_m).$$

The right side can be made as small as we please by choice first of m and then of x which proves the inclusion of Young's criterion in Lebesgue's.

§ 11

21. The case of a function having a discontinuity of the second kind was first carefully considered by Du Bois-Reymond. But the Criterion VII given in my first lecture for such a case is too difficult of proof in the course of a lecture. I shall therefore content myself by giving the proof of the criterion for the special case when

$$f(x_0+t)+f(x_0-t)=\cos\log\frac{1}{t},$$

as the treatment of this case may give you an idea of the beautiful analysis of Du Bois-Reymond.

We have to consider

$$I=\frac{1}{\pi}\int_0^{\epsilon}\left(\cos\log\frac{1}{t}\right)\frac{\sin(2n+1)t}{t}\,dt.$$

This equals

$$\frac{1}{\pi}\int_0^{(2n+1)\epsilon}\cos\left(\log\frac{2n+1}{\theta}\right).\frac{\sin\theta}{\theta}\,d\theta,$$

where $\theta=(2n+1)t.$

It is obvious that the integral

$$\int_0^\infty \begin{matrix}\cos\frac{1}{\theta}\\ \text{or}\\ \sin\frac{1}{\theta}\end{matrix} \frac{\sin\theta}{\theta}d\theta$$

exists. Calling this integral $\mathrm{R}\times \begin{matrix}\cos\gamma\\ \text{or}\\ \sin\gamma\end{matrix}$, we have, for increasing values of n, I behaving as $\frac{1}{\pi}\,\mathrm{R}\cos\{\log((2n+1)+\gamma\}$. Therefore I does not tend to any limit as n tends to infinity ; it fluctuates between $\pm\frac{\mathrm{R}}{\pi}$.

§ 12

21. I come now to refer briefly to the criteria for the uniformity of convergence in an interval.

The Criterion IX, follows from the reasoning in the proof which I have given to-day of Lebesgue's criterion.

The Criterion VIII is proved to be true ; as, if it holds, the condition of Criterion IX is satisfied.

THIRD LECTURE

Convergence Defects.

§ 13.

23. In to-day's lecture, I will try to prove the statements which I made in my first lecture with reference to the six questions (A_1), (A_2), (A_3), (A_4), (B_1) and (B_2) which I framed. Beginning with (A_1), *viz.*, Can a continuous function exist the Fourier series of which is divergent only at a point ?, I proceed to prove that the series

$$\Phi(t)=\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}\, \phi(M_\nu, m_\nu, t) \qquad \ldots \quad (1)$$

diverges at $t=o$, where, as stated in the first lecture,

$$\phi(M, m, t)=\sum_{n=1}^{m} \frac{1}{n} \cos (M+m-n+1)t - \sum_{n=1}^{m} \frac{1}{n} \cos (M+m+n)t,$$

$$m_\nu=2^{\nu^3}, \; M_1=o, \; M_\nu=2(m_1+m_2+\ldots+m_{\nu-1}).$$

The proof may be divided into four parts : I will first prove that for every positive integral value of m and every real value of M and t, $|\phi(M, m, t)| < \pi+2$; next I will prove that the series for $\Phi(t)$ is the Fourier series of a continuous function ; and then it will be proved that for $t=o$ the series is divergent ; and lastly I will show how the series is convergent at all other points.

(a) $\phi(M, m, t)=2 \sin (M+m+\frac{1}{2})t. \sum_{n=1}^{m} \frac{\sin (2n-1)\frac{t}{2}}{n}.$

But it is known* that

$$\left| \sum_{n=1}^{m} \frac{\sin (2n-1)\frac{t}{2}}{n} \right| < \frac{\pi}{2} + 1,$$

whatever m and t may be. Hence

$$| \phi | < \pi + 2.$$

(*b*) The general term of the series for Φ (t) is numerically less than $\frac{\pi+2}{\nu^2}$. But $\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}$ is absolutely convergent; therefore the series for Φ is also absolutely convergent and, consequently, uniformly convergent; hence Φ is a continuous function of t. Also the series for Φ is a Fourier series of this function. For, multiplying both the sides of (1) by cos rt, because of the uniformity of convergence, we can integrate term by term. Hence

$$\Phi = \sum_{r=0}^{\infty} c_r \cos rt, \qquad \ldots \quad (2)$$

* Consider the series $\sum_{r=1}^{\infty} \frac{\sin (2r-1)x}{r}$; it is known to represent $f(x) = \left(\frac{\pi}{2} - x\right) \cos x + \sin x \log (2 \sin x)$, $(0 < x < \pi)$. Now it is well-known that S_n, the Fejér mean of a series $\sum_{r=0}^{\infty} u_r$ is equal to $s_n - \frac{\sum_{1}^{n} r u_r}{n+1}$.

Thus

$$s_n = S_n + \frac{\sum_{1}^{\infty} r u_n}{n+1}. \quad \text{Therefore } | s_n | \leq | S_n | + \left| \frac{\sum_{1}^{n} r\, u_r}{n+1} \right|.$$

Applying this to the series for $f(x)$, we have

$$\left| s_n (x) \right| \leq \left| S_n(x) \right| + \frac{1}{n+1} \left| \sum_{1}^{n} \sin (2r-1)x \right|$$

But S_n is obviously always less than the upper limit of $f(x)$, *i.e.*, $\frac{\pi}{2}$.

Hence $\left| \sum_{r=1}^{n} \frac{\sin (2r-1)x}{r} \right| < \frac{\pi}{2} + 1$ whatever x and n may be.

where

$$c_r = \frac{1}{\nu^2(2^{\nu^3}-\lambda+1)}, \quad (\lambda=1, 2,\ldots 2^{\nu^3}),$$

$$c_r = \frac{1}{\nu^2(2^{\nu^3}-\lambda)}, \quad (\lambda=2^{\nu^3}+1,\ 2^{\nu^4}+2,\ldots 2.\ 2^{\nu^3}),$$

$$r=M_\nu+\lambda,\ 1\leq\lambda\leq M_{\nu+1}-M_\nu .$$

(*c*) Putting $t=0$ in (2) we have

$$\sum_1^\infty c_r.$$

Now the $(M_\nu+2^{\nu^3})$ *th* partial sum of this series is

$$\frac{1}{\nu^2}\left(\frac{1}{2^{\nu^3}}+\frac{1}{2^{\nu^3}-1}+\frac{1}{2^{\nu^3}-2}+\ldots+\frac{1}{2}+1\right)$$

$$>\frac{1}{\nu^2}\log\ (2^{\nu^3})\ i.e.\ \nu\log 2.$$

Therefore the series $\sum_1^\infty c_r$ cannot be convergent; for, with increasing ν, the aforesaid partial sum increases indefinitely.

(*d*) Let ϵ be an arbitrarily small quantity >0, then for $\epsilon\leq t\leq 2\pi-\epsilon$, the remainder $R_p(t)=\sum_p^\infty c_r\cos rt$ can be shown * to be numerically less

* Lemma : $|\lambda_1\cos(\alpha+1)t+\lambda_2\cos(\alpha+2)t+\ldots+\lambda_m\cos(\alpha+mt)|<\frac{\pi}{\epsilon}L$, where $\epsilon>0$ and L is *not* less than any of the λ's which increase or decrease monotonously.

This follows from the fact that $\left|\frac{1}{2}+\sum_{p=1}^{r}\cos px\right| = \left|\frac{\sin(2r+1)\frac{x}{2}}{2\sin\frac{x}{2}}\right|$

$$<\frac{1}{2\sin\frac{\epsilon}{2}}<\frac{1}{2.\frac{2}{\pi}\frac{\epsilon}{2}}=\frac{\pi}{2\epsilon}$$

If l be any positive number greater than r, similar inequality holds. Hence by subtraction $\left|\sum_{p=r+1}^{l}\cos px\right|<2.\frac{\pi}{2\epsilon}=\frac{\pi}{\epsilon}$.

than $2\pi/\epsilon(\nu_p - 1)$, where ν_p is the index of the group $\phi(M_\nu, m_\nu, t)$ from which $\cos pt$ is taken. Therefore R_p is less than any prescribed quantity if p is chosen sufficiently large. Hence convergence follows.

24. On account of its historical importance I will give the example of Du Bois-Reymond* and will also indicate his proof of the divergence of the Fourier series for that function.

Let $f(t)$ be an even function; and, for $t > o$, let it equal

$$\rho(t) \sin \Psi(t),$$

where

$$1 \succ \rho(t) \succ \frac{1}{\sqrt{\log \frac{1}{t}}},$$

and $\Psi(t)$ is defined as follows for the interval (o, t_0) :—

$$\Psi(t) = h_p\, t \text{ in the interval } (t_{p-1}, t_p),$$

where

$$h_p = \frac{1}{t_{p-1}\, t_p}, \quad t_p = \frac{t_0}{\Pi_{q=0}^{p-1} (2^q + 1)}, \quad t_0 = \frac{1}{\pi}.$$

It will be noted that, although Ψ is not monotone or continuous, $f(t)$ is continuous throughout. I proceed to show how

$$I \equiv \int_0^{t_0} f(t)\, \frac{\sin (2n+1)t}{t}\, dt$$

can be proved to be divergent when n tends to infinity.

Now apply Abel's theorem, to the inequality

$$| \cos(\alpha+1)t + \cos(\alpha+2)t + \ldots + \cos(\alpha+m)t | < \frac{\pi}{\epsilon}.$$

Then $| \lambda_1 \cos(\alpha+1)t + \ldots | \preceq \frac{\pi}{\epsilon} L.$

Again

$| R_p(t) |$ < sum of the absolute values of the positive terms in the groups and the absolute values of the negative terms in the groups; the positive terms as well as the negative terms in each group remain in absolute value less then $\frac{\pi}{\epsilon} \cdot \frac{1}{\nu^2}$

$$\therefore \quad | R_p(t) | < \frac{2\pi}{\epsilon} \left\{ \frac{1}{\nu_p^2} + \frac{1}{(\nu_p+1)^2} + \ldots \text{to inf.} \right\} < \frac{2\pi}{\epsilon} \cdot \frac{1}{\nu_p - 1}.$$

* *L. c.*, pp. 75-81.

(a) Consider

$$I_p \equiv \int_0^{t_0} f(t) \frac{\sin h_p t}{t}\, dt$$

$$= \int_0^{t_p} + \int_{t_p}^{t_{p-1}} + \int_{t_{p-1}}^{t_0}$$

and denote the above three integrals respectively by J_0, J_1, J_2,

Then $J_0 = \sum_{q=p}^{q=\infty} \int_{t_{q+1}}^{t_q} \frac{\rho(t)}{t} \sin t\, h_{q+1} \sin t\, h_p\, dt$

$$= \tfrac{1}{2} \sum_{q=p}^{\infty} \left[\frac{\rho(t_{q+1})}{t_{q+1}} \left\{ \frac{\sin t\,(h_{q+1}-h_p)}{h_{q+1}-h_p} - \frac{\sin t\,(h_{q+1}+h_p)}{h_{q+1}+h_p} \right\}_{t_{q+1}}^{\xi_q} \right.$$

$$\left. + \frac{\rho(t_q)}{t_q} \left\{ \frac{\sin t(h_{q+1}-h_p)}{h_{q+1}-h_p} - \frac{\sin t(h_{q+1}+h_p)}{h_{q+1}+h_p} \right\}_{\xi_q}^{t_q} \right]$$

by applying the second mean-value theorem.

Similarly,

$$J_2 = \sum_{q=1}^{q=p-1} \int_{t_q}^{t_{q-1}} \frac{\rho(t)}{t} \sin t\, h_q \sin t\, h_p\, dt$$

$$= \tfrac{1}{2} \sum_{q=1}^{q=p-1} \left[\frac{\rho(t_q)}{t_q} \left\{ \frac{\sin t\,(h_p-h_q)}{h_p-h_q} - \frac{\sin t\,(h_p+h_q)}{h_p+h_q} \right\}_{t_q}^{\xi_{q-1}} \right.$$

$$\left. + \frac{\rho(t_{q-1})}{t_{q-1}} \left\{ \frac{\sin t\,(h_p-h_q)}{h_p-h_q} - \frac{\sin t(h_p+h_q)}{h_p+h_q} \right\}_{\xi_{q-1}}^{t_{q-1}} \right] \quad \ldots \quad (2)$$

Also

$$J_1=\int_{t_p}^{t_{p-1}} \frac{\rho' t)\ (\sin t\, h_p)^2}{t} dt=\tfrac{1}{2} \int_{t_p}^{t_{p-1}} \frac{\rho(t)}{t} dt$$

$$-\tfrac{1}{2} \int_{t_p}^{t_{p-1}} \frac{\rho(t)}{t} \cos\ (2h_p\ t) dt\ ;$$

and, applying the second mean value theorem to

$$\int_{t_p}^{t_{p-1}} \frac{\rho(t)}{t} \cos\ (2h_p\ t) dt,$$

it is found to equal

$$\frac{\rho(h_p\ t_p)}{h_p t_p} \int_{t_p\ h_p}^{\bar{\xi}} dt \cos 2t + \frac{\rho(t_{p-1}\ h_p)}{t_{p-1}\ h_p} \int_{\bar{\xi}}^{t_{p-1}\ h_p} dt \cos 2t \ \ldots \quad (3)$$

(*b*) It is easily seen that the last expression in each of the equations (1), (2), (3) tends to zero as p tends to infinity. Let us take for example the expression in the right side of (2) ; it will suffice to prove that the expression tends to 0 even if $\rho=1$. The co-efficients of the sines have the forms

$$\frac{1}{t_q\ h_p}, \frac{1}{1\pm\frac{h_q}{h_p}}, \frac{1}{t_{q-1}\ h_p}, \frac{1}{1\pm\frac{h_q}{h_p}}.$$

Since $t_q < t_{q-1}$, $h_q < h_p$, the expression will vanish for $p=\infty$ if $\frac{p}{t_{p-1}h_p}$ vanishes.

But this is, by hypothesis, pt_p, *i.e.*, $\frac{pt_0}{\prod_0^{p-1}(2^q+1)}$, which obviously vanishes for $p=\infty$.

(*c*) Thus it is proved that, with increasing p, I_p behaves as

$$\tfrac{1}{2}\int_{t_p}^{t_{p-1}} \frac{\rho(t)}{t}\,dt$$

which is $\succ \tfrac{1}{2}\rho(t_p).\log\frac{t_{p-1}}{t_p}$, *i.e.*, $\tfrac{1}{2}\rho(t_p).\log(1+2^{p-1})$

$$\succ \tfrac{1}{2}\frac{\log(1+2^{p-1})}{\sqrt{\log\frac{1}{t_p}}},$$

since by hypothesis

$$\rho(t) \succ \frac{1}{\sqrt{\log\frac{1}{t}}}.$$

Now $$t_p = \frac{t_0}{\prod_0^{p-1}(2^q+1)};$$

hence $$\log\frac{t_0}{t_p} = \{\log(1+2^{p-1})+\log(1+2^{p-2})+\ldots$$

$$+\log(1+2)+\log 2\}$$

$$=\log\{2^{p-1}.2^{p-2}.2^{p-3}\ldots 2^1.2^0\}$$

$$+\log\left(1+\frac{1}{2^{p-1}}\right)+\log\left(1+\frac{1}{2^{p-2}}\right)+\ldots\log(1+\tfrac{1}{2}).$$

Therefore

$$I_p \succ \tfrac{1}{2}\frac{\log(1+2^{p-1})}{\sqrt{A\log 2^{\frac{p(p-1)}{2}}}},$$ where A is less than a finite

number whatever p may be.

Thus $$I_p > 1.$$

Now I cannot converge with increasing n; for if it did I_p would converge with increasing p. Therefore the Fourier series for $f(t)$ is divergent at $t=0$.

25. Before taking up question (A_2), I will give the example of Schwarz and an example of Lebesgue as particular cases of Du Bois-Reymond's example given above, and then give a second example of Lebesgue constructed on a model different from Du Bois-Reymond's or Fejér's.

(*i*) *Schwarz's example.* "Divide the interval $\left(\frac{\pi}{2}, 0\right)$ into the following sequence of diminishing sub-intervals

$$\left[\frac{\pi}{2}, \frac{\pi}{(1)}\right], \left[\frac{\pi}{(1)}, \frac{\pi}{(2)}\right], \cdots \left[\frac{\pi}{(\lambda-1)}, \frac{\pi}{(\lambda)}\right], \ldots,$$

where one has

$$(\lambda)=1.3.5.\ldots.(2\lambda+1) \text{ for } \lambda=1, 2, 3,\ldots$$

Let us consider a function $f(\beta)$ which shall be defined in the λth sub-interval by the formula $f(\beta)=c_\lambda \sin(\lambda)\beta$, where the constants c_1, $c_2,\ldots c_\mu,\ldots$form a sequence decreasing right down to 0. This function is evidently continuous; it decreases when β approaches zero by presenting an infinite number of maxima and minima with decreasing amplitudes, and we suppose that at $\beta=0$ it has the value zero." If c_μ is such that $c_\mu \log(2\mu+1)$ tends to infinity when μ tends to infinity, then the Fourier series for $f_1(\beta)$ shall diverge for $\beta=0$, where $f_1(2\beta)=f_1(-2\beta)=f(\beta)$ for $0\leq\beta\leq\pi$.

(*ii*) *Lebesgue's first example.** "Let $c_1, c_2,\ldots$be numbers tending to 0; $n_0, n_1, n_2,\ldots$being odd integers increasing indefinitely, let us put

$$a(k)=n_0 n_1 n_2 n_3 \ldots n_k,$$

and denote by I_k the interval

$$\left[\frac{\pi}{a(k-1)}, \frac{\pi}{a(k)}\right].$$

* *L.c.*, pp. 85-86.

Let us define a continuous function by the condition $f(x)=f(-x)$ and in I_k

$$\phi(t)=c_k\,[\sin a(k)t]\,\frac{\sin t}{t}\,,\ ''$$

where

$$\phi(t)=f(x+2t)+f(x-2t)-2f(x).$$

$f(x)$ is thus completely determinate for sufficiently small x and elsewhere it is defined by the condition that it is throughout continuous and of period 2π. For $c_k=\frac{1}{4^k}$, $n_k=3^{3^k}$ the Fourier series of $f(x)$ diverges at $x=0$. Generally, if c_k is such that

$$\frac{1}{2}c_k\,\log\,n_k-\sum_{p=1}^{k-1}c_p\,\log\,n_p$$

increases indefinitely with k, the series diverges at $x=0$.

*(iii) Lebesgue's second example.** " I put

$$f(x)=\epsilon_1 f_1(n_1x)+\epsilon_2 f_2(n_2x)+\epsilon_3 f_3(n_3x)+\ldots=F_p(x)+\epsilon_{p+1}f_{p+1}(n_{p+1}x)$$

$$+\epsilon_{p+2}f_{p+2}(n_{p+2}x)+\ldots\,;$$

in this expression the ϵ's are positive and the series $\Sigma\epsilon_i$ is convergent and of sum inferior to 1, the functions f_i are functions of period 2π, continuous, even, of limited variation and inferior to 1 in absolute value. Moreover, one at least of the (partial) sums of the series of Fourier of $\epsilon_i f_i(t)$ ought, for $t=0$, to exceed $i+1$ in absolute value; let this sum be the $p_i th$. Also let n_i be the smallest of the indices of the (partial) sums of the uniformly convergent Fourier series of F_{i-1} from which onwards these (partial) sums do not exceed 1 in absolute value.

In these conditions, it is evident that f is continuous and of period 2π and that, for $x=0$, the $n_i p_i th$ partial sum of the series of Fourier of f, which is equal to the corresponding sum of F_i, has an absolute value exceeding i. The series of Fourier of f diverges for $x=0$."

* *L.c.*, pp. 87-88.

§ 14.

26. The following example * due to Neder proves that the answer to (A_2), *viz.*, Can a continuous function exist the Fourier series of which is divergent *only* at the points of an enumerable and everywhere dense set ?, is in the affirmative :—

Let

$$M_1=0\,;\ M_\nu = 2\sum_{\alpha=1}^{\nu-1} m_\alpha\,,\quad m_\nu = 2^{\nu^3},\ (\nu=1,2,3,\ldots)\,;$$

further let

$$\phi_\nu(t) = \sum_{k=M_\nu+1}^{M_\nu+1} \frac{\cos kt}{M_\nu + m_\nu + \frac{1}{2} - k},\ (\nu=1,\ 2,\ 3,\ldots).$$

Then consider

$$\phi(t)=\sum_{\nu=1}^{\infty} \frac{\epsilon_\nu}{\nu^2}\phi_\nu\,(t-t_\nu\,), \qquad \ldots\ (1)$$

where, for $\nu=1, 2, 3,\ldots$, t_ν is an arbitrary real number, and $0 \leq \epsilon_\nu \leq 1$. Now it can be easily proved that an absolute constant Γ exists such that

$$|\phi_\nu\ (t)|\leq\Gamma$$

whatever positive integral value ν may have and whatever real value t may have. Therefore $\phi(t)$ converges absolutely, and, consequently, uniformly. Hence it represents a function of t which has period 2π and which is continuous for all real values of t.

Now let $\{\theta_\rho\}$ be a sequence, of unequal numbers, which is everywhere dense in the interval $(0, 2\pi)$; and form a new sequence

$$\theta_1\ ;\ \theta_1, \theta_2\ ;\ \theta_1, \theta_2, \theta_3\ ;\ \theta_1, \theta_2, \theta_3, \theta_4\ ;\ldots\ldots$$

in which each θ_ρ occurs infinitely often.

* *L. c.*, pp. 264-265.

Also define a positive integral number $p(\nu)$ by the equation

$$p(\nu)=\nu-{}^{n}C_2 \quad \text{for} \quad {}^{n}C_2<\nu\leq{}^{n+1}C_2, \quad (n=1, 2, 3,\ldots);$$

and consider the sequence

$$\theta_{p(\nu)}$$

for $\nu=1, 2, 3,\ldots$.

Put in (1),

$$\epsilon_\nu=\frac{1}{p(\nu)}, \quad t_\nu=\theta_{p(\nu)};$$

then

$$\phi(t)=\sum_{\nu=1}^{\infty}\frac{1}{\nu^2\cdot p(\nu)}\sum_{k=M_\nu+1}^{k=M_\nu+1}\frac{\cos k\left(t-\theta_{p(\nu)}\right)}{M_\nu+m_\nu+\frac{1}{2}-k}$$

is a continuous function whose Fourier series diverges at every point of the enumerable and everywhere dense set $\{\theta_\rho\}$, and is convergent at all the remaining points of the interval $(0, 2\pi)$.

§ 15.

27. Let us consider

$$F(t)\equiv\sum_{1}^{\infty}\omega_\nu(t)\equiv\sum_{\nu=1}^{\infty}\gamma_\nu\sum_{k=M_\nu+1}^{M_\nu+1}\frac{\cos k(t-t_\nu)}{M_\nu+m_\nu+\frac{1}{2}-k},$$

where m_ν and M_ν have the same meanings as before and $\gamma_\nu=\frac{1}{\nu^2}$; also $\{t_\nu\}$ is an arbitrary but everywhere dense set in the interval $(o, 2\pi)$. It will be shown how Steinhaus * proved that the aforesaid series diverges at the points of a set which has the potency of the continuum.

* See Neder's *Dissertation*, Chapter IV, Art. 7.

Proof :

(*a*) Among the numbers t_ν for which $\nu \geq 2 \cdot 1$, there is always one t_{ν_0} and another t_{ν_1} respectively in the first and last third of an interval (a, b), $o \leq a \leq b < 2\pi$. But it can be proved without difficulty that $\omega_{\nu_0}(t_{\nu_0}) \geq \nu_0 \log 2$, $\omega_{\nu_1}(t_{\nu_1}) \geq \nu_1 \log 2$ so that each of the two $\geq 2 \log 2 > 1$. Therefore, on account of the continuity of $\omega_{\nu_0}(t)$ and $\omega_{\nu_1}(t)$, there is a sub-interval I_0 of the first third and a sub-interval I_1 of the last third of (a, b) in which $\omega_{\nu_0}(t) \geq 1$ and $\omega_{\nu_1}(t) \geq 1$ respectively. In the same manner, with the help of the numbers t_ν for which $\nu \geq 2.2$ and ν is different from ν_0 and ν_1, we can obtain intervals I_{00}, I_{01} in the first and last thirds of I_0 and intervals I_{10}, I_{11} in the first and last thirds of I_1 so that, for $\nu = \nu_{00}, \nu_{01}, \nu_{10}, \nu_{11}$, $\omega_\nu(t) \geq 2$ in these four intervals respectively. Generally, corresponding to each sequence of binary digits $\alpha_1 \alpha_2 \ldots \alpha_\lambda$, $(\lambda = 1, 2, 3 \ldots)$ there is an interval $I_{\alpha_1 \alpha_2 \ldots \alpha_\lambda}$ and a number $\nu_{\alpha_1 \ldots \alpha_\lambda}$ such that in this interval throughout $\omega_{\nu_{\alpha_1 \ldots \alpha_\lambda}}(t) \geq \lambda$. It may be noted that $I_{\alpha_1 \ldots \alpha_\lambda}$ contains each of the two intervals $I_{\alpha_1 \ldots \alpha_\lambda 0}$ and $I_{\alpha_1 \ldots \alpha_\lambda 1}$ and these lie respectively in the first and last thirds of $I_{\alpha_1 \ldots \alpha\lambda}$ quite separate from each other.

(*b*) Now to each infinite sequence of binary digits $\alpha_1 \alpha_2 \ldots$ corresponds a sequence of intervals I_{α_1}, $I_{\alpha_1 \alpha_2}$, ... of which each is contained in the preceding ; hence a point T exists which is common to all the intervals and which with I_{α_1} is contained in (a, b). This point is a point of divergence ; for, because of $\omega_{\nu_{\alpha_1 \ldots \alpha_\lambda}}(t) \geq \lambda$, the numbers $\omega_\nu(T)$, $\nu = 1, 2, \ldots$, are unbounded and, consequently, the partial sums of F(T) are also unbounded.

(*c*) Lastly, the set of points T of (a, b) has the potency of the continuum. For, corresponding to different sequences $\alpha_1 \alpha_2 \ldots, \beta_1 \beta_2, \ldots$ of binary digits, there are different numbers T ; since, if λ be the least number for which $\alpha_\lambda \neq \beta_\lambda$, of the two numbers T one belongs to $I_{\alpha_1 \ldots \alpha_{\lambda-1} 0}$ and the other to $I_{\alpha_1 \ldots \alpha_{\lambda-1} 1}$ and these sub-intervals are quite separate.

[To prove that $\omega_\nu(t_\nu) \geq \gamma_\nu \log m_\nu$, note that $\omega_\nu(t_\nu)$ is by definition equal to $\gamma_\nu \sum_{r=1}^{m_\nu} \frac{1}{r-\frac{1}{2}}$ *i.e.* $\omega_\nu(t_\nu) > \gamma_\nu \sum_{r=1}^{m_\nu} \frac{1}{r} > \gamma_\nu \log m_\nu$.

Hence $\omega_\nu(t_\nu) > \nu \log 2$.]

28. (*a*) The example of Fejér, considered in Art. 23, can be used easily to construct a continuous function whose Fourier series is divergent *at least* at the points of an enumerable and everywhere dense set. In fact,

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \phi(M_\nu, m_\nu, \nu ! t) \tag{1}$$

is such a function. For, on account of the series being absolutely and consequently, uniformly convergent, it represents a continuous function which has period 2π. Again, as in Art. 23, it follows that (1) is the Fourier series of the function. It remains only to show that at every point

$$t = \frac{m\pi}{n},$$

m and n being integral, the series diverges. This will be obvious from the fact that (1) equals, from a given $\frac{m\pi}{n}$,

$$\sum_{\nu=1}^{\nu_1} + \sum_{\nu=\nu_1}^{\infty},$$

where the first series is finite and the second is divergent if ν_1 is so large that $\pi\left(\nu_1 ! \frac{m}{n}\right)$ is an even multiple of π. Thus at every rational point the series is proved to be divergent.

(*b*) At the end of his famous memoir,* Du Bois-Reymond indicated how a series of quantities μ_p can be chosen so that the Fourier series of the function

$$F(t) = \sum_{p=1}^{\infty} \mu_p \rho(\sin pt) \cos \Psi(\sin pt)$$

* *L. c.* pp. 101-102.

may be divergent at a point of every sub-interval ever so small ; in the above ρ and Ψ have the same significance as in Art. 24. Hobson, treating Schwarz's example, gave a function * similar to $F(t)$.

(*c*) One criticism of the example of Fejér and also of the examples of Du Bois-Reymond and Hobson, if these latter are correct, is that the points of divergence must, according to the theorem of Steinhaus given in the first lecture, have the *potency of the continuum*. A second criticism was made by Neder † against the examples of Du Bois-Reymond and Hobson, that the reasoning used by the two authors is open to objection. For want of time, I need not enter into the details of Neder's objection.

§ 16.

29. As regards (A_4), *viz.*, Can a continuous function exist the Fourier series of which is divergent at every point ?' I stated in the first lecture that the question remains open ; it has not been answered as yet in an authoritative manner. It will interest you to hear the actual words of various authorities in this connection.

Lebesgue says : " In a note in the *Comptes Rendus* (December 29, 1902), M. Stekloff has indicated that the answer to the first question [*i. e.* (A_4)] should be in the negative. The proof has not yet been published and the reasoning of the note is, it appears to me, insufficient to enable one to build up this proof." (Lecons sur les séries trigonométriques, 1906, p. 89, footnote.)

Fejér's opinion is : " Du Bois-Reymond has framed the question, whether an everywhere continuous function exists whose Fourier series everywhere diverges. This problem remains unsolved up to to-day " *Crelle's Journal*, Bd. 138, 1910, p. 52).

Neder states : The question whether a continuous function exists whose Fourier series diverges everywhere in the interval $(0, 2\pi)$ remains open." (*Inaugural Dissertation*, Göttingen, 1919, p. 26.)

Hobson says : It is not definitely known whether a Fourier series corresponding to a continuous function, can be such that the series fails to converge at every point of an interval. " (*Theory of Functions*, 1907, p. 701.)

* *Proc., L. M. S.*, Ser. 2, Vol. 3, 1905 ; also " Theory of Functions of a Real Variable," 1st edition, pp. 705-707.

† " Ueber stetige Funktionen mit ueberalldicht divergierender Fourierreihe " (*Jahresbericht d. deutschen Math.—Ver.*, Bd. 30, 1921, pp. 153-155).

The function constructed by Kolmogoroff * for which the Fourier series fails to converge at the points of a set of measure 2π, is not continuous and neither is its square summable in the interval $(0, 2\pi)$.

§ 17.

30. I will conclude this lecture by proving that the questions (B_1) and (B_2) should be answered in the affirmative.

Taking up first (B_1), *viz.*, Can a continuous function exist the Fourier series of which for $(0 \leq x < 2\pi)$ is convergent at every point without exception but not uniformly convergent in any subinterval including 0 ? , consider the function:

$$\Psi(t) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \psi(M_\nu, m_\nu, t), \qquad \ldots (2)$$

where

$$\psi(M, m, t) = \sum_{n=1}^{m} \frac{1}{n} \sin (M+m-n+1)t - \sum_{n=1}^{m} \frac{1}{n} \sin (M+m+n)t,$$

and m_ν, M_ν have the same meanings as in Art. 23.

Then $\Psi(t)$ is a continuous function answering the requirements of (B_1).

Proof:

As in Art. 23, $|\psi(M, m, t)| < \pi + 2$, m having any positive integral value and M, t having any real values. Hence the series in (2) is absolutely, and, consequently, uniformly convergent; therefore $\Psi(t)$ is a continuous function with period 2π. Reasoning as in Art. 23, we find that the series in (2) is the Fourier series

$$\sum_{1}^{\infty} c_r \sin r t$$

of $\Psi(t)$, c_r having the same values as in Art. 23.

(*i*) Now the above series converges for every value of t including 0.

* *Fundamenta Math.*, Vol. 4 (1923), p. 324.

For, the remainder $R_p(t)$ after $(p-1)$ terms, *i.e.*, $\sum_{r=p}^{\infty} c_r \sin rt$, is numerically less than

$$\frac{2\pi}{\epsilon}\left\{\frac{1}{\nu_p^{\,2}}+\frac{1}{(\nu_{p+1})^2}+\ldots\ldots\text{to inf.}\right\}^{*} \quad \text{if} \quad \epsilon>0 \text{ and } \epsilon\leq t\leq 2\pi-\epsilon,$$

ν_p being the index of the group of terms $\psi(M_\nu, m_\nu, t)$ from which $\sin pt$ is taken. Therefore $|R_p| < \frac{2\pi}{\epsilon}\cdot\frac{1}{\nu_p-1}$, which becomes less and less if p is increased. Therefore the series converges for $\epsilon\leq t\leq 2\pi-\epsilon$; at 0 it obviously converges.

(*ii*) Again the series cannot be uniformly convergent in any sub-interval (o, ϵ). For, suppose, if possible that a number P exists such that it corresponds to a given quantity $\delta>0$ so that

$$|R_p(t)| < \delta,\ p\geq P,\ 0\leq t\leq\epsilon.$$

If this were true, by taking p_1 and p_2 so large that $R_{p_2}(t)-R_{p_1}(t)$ is the sum of the 2^{ν^3} positive terms given by putting in $\sum_{1}^{\infty} c_r \sin rt$

$$r=M_\nu+\lambda,\ (\lambda=1, 2, \ldots\ldots 2^{\nu^3}),$$

we have

$$R_{p_2}-R_{p_1}=\left|\sum_{r=M_\nu+1}^{M_\nu+2^{\nu^3}} c_r \sin\frac{r\pi}{2(M_\nu+2^{\nu^3})}\right|\leq 2\delta,$$

because $t=\frac{\pi}{2(M_\nu+2^{\nu^3})}$ is in $(0, \epsilon)$ if ν is sufficiently large.

* The proof is similar to that given in Art. 23 under the footnote to (*d*).

But it is easy to prove * that the finite series given above is numerically greater than $\frac{1}{\sqrt{2}} \log 2. \frac{\nu^3-1}{\nu^2}$ which tends to infinity with ν and is certainly not less than 2δ.

31. The last question is (B_2), *viz.*, Can a continuous function exist the Fourier series of which for $0 \leq x < 2\pi$ is convergent at every point without exception but is not uniformly convergent in any subinterval of $(0, 2\pi)$?

I proceed to prove, by giving Neder's example, † that this question should be answered in the affirmative.

* Consider

$$\frac{\sin (M+1)x}{n} + \frac{\sin (M+2)x}{n-1} + \ldots + \frac{\sin (M+n)x}{1}$$

$(n=1, 2, 3.\ldots\ldots)$, $M>0$,

when

$$x = \frac{\pi}{2(M+n)}. \text{ Now } (M+\nu)\frac{\pi}{2(M+n)} \leq \frac{\pi}{2} \text{ if } \nu = 1, 2, \ldots n.$$

Therefore

$$\sum_{\nu=1}^{n} \frac{\sin (M+\nu).\frac{\pi}{2(M+n)}}{n-\nu+1} > \sum_{\nu=\left[\frac{n}{2}\right]+1}^{n} \frac{\sin (M+\nu)\frac{\pi}{2(M+n)}}{n-\nu+1}, \text{ where}$$

$\left[\frac{n}{2}\right]$ is the greatest integer in $\frac{n}{2}$. But for the latter series

$$\frac{\pi}{4} < (M+\nu)\frac{\pi}{2(M+n)} \leq \frac{\pi}{2}.$$

Therefore the first series in the above $> \sin\frac{\pi}{4} \sum_{\nu=\left[\frac{n}{2}\right]+1}^{n} \frac{1}{n-\nu+1}$

$$= \frac{1}{\sqrt{2}}\left\{1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\ldots+\frac{1}{n-\left[\frac{n}{2}\right]}\right\} \geq (1/\sqrt{2})\left(1+\tfrac{1}{2}+\ldots+\frac{1}{\left[\frac{n}{2}\right]}\right)$$

$$> \frac{1}{\sqrt{2}} \log.\left[\frac{n}{2}\right].$$

Applying this to the present case we have $R_{p_2} - R_{p_1}$ greater than

$$\frac{1}{\nu^2} \times \frac{1}{\sqrt{2}}. \log \left(\frac{2^{\nu^3}}{2}\right) i.e., \frac{1}{\sqrt{2}} \log 2. \frac{\nu^3-1}{\nu^2}.$$

† *L. c.*, pp. 267-268.

Let θ_ρ and $p(\nu)$ have the same meanings as in Art. 26. Then put

$$\epsilon_\nu = \frac{1}{p(\nu)}, \; t_\nu = \theta_{p(\nu)} + \frac{1}{\nu}, \qquad (\nu = 1, 2, 3 \ldots\ldots)$$

Thus

$$\phi(t) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^2 p(\nu)} \left\{ \sum_{k=M_\nu+1}^{M\nu+1} \frac{\cos k\left(t - \theta_{p(\nu)} - \frac{1}{\nu}\right)}{M_\nu + m^\nu + \frac{1}{2} - k} \right\}$$

$$(\nu = 1, 2, 3, \ldots\ldots)$$

is a continuous function of period 2π which converges at every point in the interval $(0, 2\pi)$ and is still non-uniformly convergent in every sub-interval of $(0, 2\pi)$.

FOURTH LECTURE

Cesàro Summability.

§ 18.

32. I will discuss to-day the Cesàro summability of various orders and begin by giving in the words of Prof. Fejér his enunciation of his famous result for the summability (C1) of the Fourier series of a continuous function. Fejér * says : " Let it remain undecided whether continuous functions exist whose Fourier series diverge for every x (as for example, continuous functions exist which have not any differential coefficient at any point)—in any case we can say : from the mere continuity of the function at a point x, does not follow the convergence of the Fourier series at this point. There must be further certain restrictions imposed on the mode of the continuity—if this phrase is allowed—in order that the convergence of the series be assured. Whilst we leave aside the question for the sufficient conditions for the convergence of the Fourier series, a question which has been somewhat too fully treated in the literature on the subject of Fourier series, we turn to the Fourier series by putting a question which completely corresponds to the question of Borel and Mittag-Leffler in reference to the power series of a complex variable and which is as follows : Is it possible to derive, from the Fourier series corresponding to a continuous function $f(x)$, *e.g.*, from the sequence

$$s_0(x),\ s_1(x),\ldots s_n(x) \qquad \ldots \qquad (1)$$

which is equivalent to the series, another sequence which converges for any arbitrarily chosen value of x and indeed to $f(x)$ as the limit ? This is indeed possible. One need go from the sequence (1) to the sequence of arithmetic means

$$s_0(x),\ \frac{s_0(x)+s_1(x)}{2},\ \ldots\ \frac{s_0(x)+\ldots+s_{n-1}(x)}{n},\quad \ldots \qquad (2)$$

and then this new sequence, which consists as much of finite trigonometric series as the sequence (1), possesses the desired property."

* " Untersuchungen über Fouriersche Reihen" (*Math. Ann.*, Bd. 58, 1904, Seiten 51-52).

33. The above statement of Fejér, *viz.*, that

$$S_n(x) \equiv \frac{s_0 + s_1 + \ldots + s_{n-1}}{n} \qquad (T_3)$$

tends to $f(x)$ with increasing n, may be proved as follows :

(a) First, as according to (1) of Art. 17,

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}} f(x+t)\,dt,$$

$$S_n(x) = \frac{1}{n\pi}\int_{-\pi}^{\pi} \frac{\sum_0^{n-1} \sin(2r+1)\frac{t}{2}}{2\sin\frac{t}{2}} f(x+t)dt$$

$$= \frac{1}{2n\pi}\int_{-\pi}^{\pi} \left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 f(x+t)dt$$

$$= \frac{1}{n\pi}\int_{0}^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 \{f(x+2z)+f(x-2z)\}dz,$$

putting $2z$ for t.

Therefore, since*

$$\operatorname*{Lim}_{n=\infty} \frac{1}{n\pi}\int_{0}^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 dz = \frac{1}{2},$$

* Putting $f(x)=1$, we have $s_0=1$, $s_n=0$ for $n>0$.

Hence $S_n = \frac{n}{n} = 1$ for every n.

Therefore $\frac{2}{n\pi}\int_0^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 dz = 1$ for every n ; and, consequently,

$$\operatorname*{Lim}_{n=\infty} \frac{1}{n\pi}\int_{0}^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 dz = \tfrac{1}{2}.$$

we have to prove that $S_n(x)-f(x)$, *i.e.*

$$\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin nz}{\sin z}\right)^2\phi(z)dz \qquad \ldots \quad (1)$$

tends to 0 with increasing n, $\phi(z)$ standing for

$$f(x+2z)+f(x-2z)-2f(x).$$

(*b*) Next, consider (1) in the form

$$\frac{1}{n\pi}\int_a^{\frac{\pi}{2}}\left(\frac{\sin nz}{\sin z}\right)^2\phi(z)dz+\frac{1}{n\pi}\int_0^{a}\left(\frac{\sin nz}{\sin z}\right)^2\phi(z)dz\ ,$$

where $a>0$ is chosen to be arbitrarily small but independent of n.

Then, since

$$\frac{1}{2n\pi}\int_a^{\frac{\pi}{2}}\left(\frac{\sin nz}{\sin z}\right)^2\phi(z)dz$$

$$=\frac{1}{4n\pi}\int_a^{\frac{\pi}{2}}\frac{\phi(z)}{(\sin z)^2}dz-\frac{1}{4n\pi}\int_a^{\frac{\pi}{2}}\frac{\phi(z)}{(\sin z)^2}\cos 2nz\,dz,$$

and $\frac{\phi(z)}{(\sin z)^2}$ is integrable in the interval $\left(a, \frac{\pi}{2}\right)$, it follows that, with increasing n, the first integral on the right tends to 0 and that by the theorem of Art. 16 the second also tends to 0. Therefore we have to prove that

$$\operatorname*{Lim}_{n=\infty}\frac{1}{2n\pi}\int_0^{a}\left(\frac{\sin nz}{\sin z}\right)^2\phi(z)dz=0 \qquad \ldots \quad (T_4)$$

(c) Since $\phi(z)$ is continuous round $z=0$, corresponding to any quantity $\delta > 0$ but as small as we please, a can be found such that

$$| \phi(z) | < \delta$$

for every value of z in the interval $(0, a)$.

Therefore

$$\left| \frac{1}{2n\pi} \int_0^a \left(\frac{\sin nz}{\sin z} \right)^2 \phi(z) dz \right| < \delta \cdot \frac{1}{2n\pi} \int_0^a \left(\frac{\sin nz}{\sin z} \right)^2 dz.$$

Therefore by making n sufficiently large the above expression can be made as small as we please. Thus it is proved that

$$\operatorname*{Lim}_{n=\infty} S_n(x) = f(x).$$

34. It will be convenient to deduce, from (1) and T_4 of the preceding article, Lebesgue's criterion,* writing for $\phi(z)$ not

$$f(x+2z)+f(x-2z)-2f(x)$$

but

$$f(x+2z)+f(x-2z)-2S,$$

where S is the limit to which S_n converges.

Now

$$\frac{1}{2n\pi} \int_0^a \left(\frac{\sin nz}{\sin z} \right)^2 \phi(z) dz = \frac{1}{2n\pi} \int_0^{\frac{\pi}{2n+1}} \left(\frac{\sin nz}{\sin z} \right)^2 \phi(z) dz$$

$$+ \frac{1}{2n\pi} \int_{\frac{\pi}{2n+1}}^a \left(\frac{\sin nz}{\sin z} \right)^2 \phi(z) dz$$

* *Math. Ann.*, Bd. 61, pp. 274, also *Lecons*, pp. 94-96.

By replacing $\frac{\sin nz}{\sin z}$ by n in the first integral on the right and sin nz by 1 in the second, we have, denoting $\int_0^z |\phi(z_1)|\, dz_1$ by $\Phi(z)$,

$$\left|\frac{1}{2n\pi}\int_0^{\alpha}\left(\frac{\sin nz}{\sin z}\right)^2\phi(z)dz\right| < \frac{n}{2\pi}\,\Phi\left(\frac{\pi}{2n+1}\right) + \frac{\pi}{8n}\int_{\frac{\pi}{2n+1}}^{\alpha}\frac{1}{z^2}\,|\phi(z)|\,dz.$$

But, because of $\Phi'(0)$ being zero, the first part of the above tends to zero ; denoting the second by J_2, we have by integration by parts

$$\frac{4}{\pi^2}J_2 \leq \frac{1}{2n\pi}\left[\Phi(z).\frac{1}{z^2}\right]_{\frac{\pi}{2n+1}}^{\alpha} + \frac{1}{n\pi}\int_{\frac{\pi}{2n+1}}^{\alpha}\frac{\Phi(z)}{z^3}\,dz.$$

As with increasing n, the first part of the above tends to zero, we have to consider only

$$J_3 \equiv \frac{1}{n\pi}\int_{\frac{\pi}{2n+1}}^{\alpha}\frac{\Phi(z)}{z^3}\,dz.$$

Now, because of $\Phi'(0)$ being zero, corresponding to a small quantity $\epsilon > 0$, it is always possible to choose a second quantity δ so that $\frac{\Phi(z)}{z} < \epsilon$ for $z \leq \delta$. Therefore, when n is sufficiently large,

$$J_3 < \frac{\epsilon}{n\pi}\int_{\frac{\pi}{2n+1}}^{\delta}\frac{dz}{z^2} + \frac{\Phi(\alpha)}{n\pi}\int_{\delta}^{\alpha}\frac{dz}{z^3} < \epsilon + \frac{\Phi(\alpha)}{2n\pi\delta^2} < 2\epsilon\,;$$

hence J_3 tends to 0 with $\frac{1}{n}$.

35. I proceed now to verify the theorem of Fejér for the case of the series

$$\sum_{r=0}^{\infty} c_r \cos rt,$$

which corresponds to Fejér's continuous function

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \phi(M_\nu , m_\nu , t)$$

studied in Art. 23. As stated there,

$$c_r = \frac{1}{\nu^2(2^{\nu^3} - \lambda + 1)}, \quad (\lambda = 1, 2, \ldots 2^{\nu^3}),$$

$$c_r = \frac{1}{\nu^2(2^{\nu^3} - \lambda)}, \quad (\lambda = 2^{\nu^3}+1, 2^{\nu^3}+2, \ldots 2.2^{\nu^3});$$

and the series diverges at $t=0$. I will prove that the arithmetic mean S_n of the partial sums s_0, s_1, etc., of the series $\sum_{\nu=0}^{\infty} c_r$ is convergent, the limit being 0; the proof given being due to Fejér.*

Proof:

Let

$$n = m_1 + m_2 + \ldots + m_\nu + r, \quad (0 \leq r < m_{\nu+1})$$

and consider $S_n \equiv \frac{s_1 + s_2 + \ldots + s_n}{n}$.

Then, since, as shown at the end of this proof, the Lemma holds that for the series

$$\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \ldots + \tfrac{1}{2} + \tfrac{1}{1} - \tfrac{1}{1} - \tfrac{1}{2} - \tfrac{1}{3} - \ldots - \frac{1}{k-1} - \frac{1}{k} + 0 + 0 + \ldots$$

to inf. every arithmetic mean S_l is less than 2 whatever k or l may be,

$$S_n \equiv \frac{s_1 + s_2 + \ldots + s_n}{m_1 + m_2 + \ldots + m_\nu + r}$$

$$= \frac{(s_1 + s_2 + \ldots + s_{m_1}) + (s_{m_1+1} + \ldots + s_{m_1+m_2}) + \cdot}{m_1 + m_2 + \ldots + m_\nu + r}$$

* *Ann. d. l'école normale*, 1911, pp. 94-99.

$$= \frac{m_1 \cdot \dfrac{s_1+s_2+\ldots+s_{m_1}}{m_1} + m_2 \cdot \dfrac{s_{m_1+1}+\ldots+s_{m_1+m_2}}{m_2} + \ldots}{m_1+m_2+\ldots+m_\nu+r}$$

$$< \frac{\dfrac{2m_1}{1^2}+\dfrac{2m_2}{2^2}+\ldots+\dfrac{2m_\nu}{\nu^2}+\dfrac{2r}{(\nu+1)^2}}{m_1+m_2+\ldots+m_\nu+r},$$

But it is known from Algebra * that the last expression tends to 0 when ν tends to infinity, *i.e.*, when n tends to infinity. Therefore it is proved that $\lim_{n=\infty} S_n = 0$.

[*Proof of the Lemma.* For any series $\sum_1^\infty u_p$, $nS_n = \sum_1^n (n-p+1)u_p$.

Hence

(*a*) If $1 \leq n \leq k$, $S_n = \left(\dfrac{n}{k} + \dfrac{n-1}{k-1} + \ldots + \dfrac{n-k+1}{n-k+1} \right) / n$

$$< \frac{\dfrac{k}{k}+\dfrac{k-1}{k-1}+\ldots}{n} < \frac{n}{n}, \text{ i.e., } 1;$$

(*b*) If $k < n \leq 2k$, say $n = k+q$, where $0 < q \leq k$, then

$$S_{k+q} = \frac{1}{k+q}\left\{\frac{k+q}{k}+\frac{k+q-1}{k-1}+\ldots+\frac{q+1}{1} - \frac{q}{1}-\frac{q-2}{2}\ldots-\frac{1}{q}\right\}$$

$$= \frac{1}{k+q}\left\{\sum_{m=1}^{k}\frac{q+m}{m} - \sum_{m=1}^{q}\frac{q-m+1}{m}\right\}$$

$$= \frac{1}{k+q}\left\{\sum_{m=1}^{q}\frac{q+m}{m} - \sum_{m=1}^{q}\frac{q-m+1}{m} + \sum_{m=q+1}^{k}\frac{q+m}{m}\right\}$$

* See *e.g.*, Hobson's *Theory of Functions*, Vol. 2, p. 8, Ex. 3.

$$= \frac{1}{k+q}\left\{ \sum_{m=1}^{q} \frac{2m-1}{m} + \sum_{m=q+1}^{k} \frac{q+m}{m} \right\}$$

$$< \frac{1}{q+k} \{2q+2(k-q)\} = \frac{2k}{k+q} < 2;$$

(*c*) If $2k<n<\infty$, *i.e.*, $n=k+q$ where $k<q<\infty$, then

$$S_n = \frac{1}{k+q}\left\{ \frac{k+q}{k} + \frac{k+q-1}{k-1} + \ldots + \frac{q+1}{1} - \frac{q}{1} - \frac{q-1}{2} \ldots - \frac{q-k+1}{k} \right\}$$

$$= \frac{1}{k+q}\left\{ \sum_{m=1}^{k} \frac{q+m}{m} - \sum_{m=1}^{k} \frac{q-m+1}{m} \right\} = \frac{1}{k+q} \sum_{m=1}^{k} \frac{2m-1}{m}$$

$$< \frac{2k}{k+q} < \frac{2k}{2k}, \text{ i.e., } 1.\Big]$$

§ 19.

36. I have, as already stated in the first lecture, proved the failure of Lebesgue's criterion for an important class of functions, *viz.*, those for which

$$f(x+2t)+f(x-2t)=\chi(t)\cos\psi\,(t),$$

χ and ψ being in the neighbourhood of $t=0$ monotone and ψ unbounded. To give you an idea of the proof, I content myself with the consideration of the special case in which $\chi=1$, $\psi\,(t)=\frac{1}{t}$. I will prove that in this case $\Phi'(0)$ is *not* zero and that, notwithstanding this, $\underset{n=\infty}{\text{Lim}}\, S_n(x)$ exists and is 0.

Proof:

(*a*) Consider $\Phi(z) \equiv \int_0^z \left| \cos\frac{1}{t} \right| dt$; and let

$$t_r \leq z < t_{r-1}, \qquad (1)$$

where

$$\frac{1}{t_r}=\left(r+\frac{1}{2}\right)\pi\,,$$

r being a positive integer.

Remembering that in the interval (t_r, z)

$$\left|\cos\frac{1}{t}\right| =(-1)^r\cos\frac{1}{t}$$

and, generally, in the interval (t_{r+m}, t_{r+m-1}),

$$\left|\cos\frac{1}{t}\right| =(-1)^{m+r}\cos\frac{1}{t}\,,$$

we have

$$\int_0^z\left|\cos\frac{1}{t}\right|dt=\left\{(-1)^r\int_{t_r}^z\cos\frac{1}{t}\,dt+(-1)^{r+1}\right.$$

$$\left.\times\int_{t_{r+1}}^{t_r}\cos\frac{1}{t}\,dt+(-1)^{r+2}\int_{t_{r+2}}^{t_{r+1}}\cos\frac{1}{t}\,dt+\ldots\text{to inf.}\right\}$$

$$=(-1)^r\left\{\int_{t_r}^z\cos\frac{1}{t}\,dt-\int_{t_{r+1}}^{t_r}\cos\frac{1}{t}\,dt+\int_{t_{r+2}}^{t_{r+1}}\cos\frac{1}{t}\,dt-\ldots\right\}.$$

Now

$$\frac{d}{dt}\left(t^2\sin\frac{1}{t}\right)=2t\sin\frac{1}{t}-\cos\frac{1}{t}\,,$$

and, consequently,

$$\int\cos\frac{1}{t}\,dt=\int 2t\sin\frac{1}{t}\,dt-t^2\sin\frac{1}{t}\,,$$

Therefore

$$\Phi(z)=(-1)^r\left[\left\{\int_{t_r}^{z}2t\sin\frac{1}{t}\,dt-\left(t^2\sin\frac{1}{t}\right)_{t_r}^{z}\right\}\right.$$

$$-\left\{\int_{t_{r+1}}^{t_r}2t\sin\frac{1}{t}\,dt-\left(t^2\sin\frac{1}{t}\right)_{t_{r+1}}^{t_r}\right\}$$

$$\left.+\left\{\int_{t_{r+2}}^{t_{r+1}}2t\sin\frac{1}{t}\,dt-\left(t^2\sin\frac{1}{t}\right)_{t_{r+2}}^{t_{r+1}}\right\}-\ldots\text{ to infinity}\right]. \qquad (2)$$

Now, for $t=t_{r+m}$, $(-1)^{r+m}\sin\frac{1}{t}=1$.

Therefore the equation (2) becomes *

$$\Phi(z)=\eta\int_0^z 2t\left|\sin\frac{1}{t}\right|dt-z^2\eta_1\left|\sin\frac{1}{z}\right|$$

$$+2\{t_r^2+t^2{}_{r+1}+\ldots\text{to infinity}\},$$

where η is numerically less than 1 and η_1 is ± 1.

* I take this opportunity to point out a slight error in the expression (A) of my paper "On the failure of Lebesgue's criterion for the summability (C1) of the Fourier series of a function at a point where the function has a discontinuity of the second kind." The error, *which does not affect the conclusion*, is that η and η_1 have been left out as the factors of the third and first parts respectively of the expression (A).

Similar remark applies to the expression (B).

Now, obviously, the parts contributed to $\frac{\Phi(z)}{z}$ by the first two terms in the above expression each tend to 0 with z. Therefore

$$\Phi'(0)=\operatorname*{Lim}_{z=0}\frac{2}{z}\left\{t_r^2+t_{r+1}^2+\ldots\text{to infinity}\right\}.$$

But

$$t_r^2+t^2{}_{r+1}+\ldots=\frac{1}{\pi^2}\left\{\frac{1}{(r+\frac{1}{2})^2}+\frac{1}{(r+\frac{3}{2})^2}+\ldots\right\}$$

$$=\frac{1}{\pi^2}\left\{\frac{1}{r-\frac{1}{2}+k}\right\}, \text{ where } 0<k<1, \text{ because the sum of the series}$$

inside the crooked brackets is less than $\frac{1}{r-\frac{1}{2}}$ and greater than $\frac{1}{r+\frac{1}{2}}$.

Also, on account of (1),

$$\frac{1}{t_r}\geq\frac{1}{z}>\frac{1}{t_{r-1}},$$

i.e., $\frac{1}{z}=(r-\frac{1}{2}+k_1)\pi$,

where $0<k_1\leq 1$.

Therefore

$$\Phi'(o)=\operatorname*{Lim}_{r=\infty}\frac{2}{\pi^2}\cdot\frac{(r-\frac{1}{2}+k_1)\pi}{(r-\frac{1}{2}+k)}=\frac{2}{\pi}.$$

(*b*) For the convergence of $S_n(x)$ to S, it is necessary and sufficient that

$$\operatorname*{Lim}_{n=\infty}\frac{1}{2n\pi}\int_0^{\alpha}\phi(z)\left(\frac{\sin nz}{\sin z}\right)^2 dz=0,$$

$\alpha>0$ being arbitrarily small but independent of n ; this is theorem (T_4).

The condition is equivalent to the condition

$$\lim_{n=\infty} \frac{1}{2n\pi} \int_0^a \phi(z) \cdot \left(\frac{\sin nz}{z}\right)^2 dz = 0.$$

Now, integrating by parts, and denoting $\int_0^z \phi(t)dt$ by $\Phi(z)$,

$$\int_0^a \phi(z) \left(\frac{\sin nz}{z}\right)^2 dz = \Phi(a) \cdot \left(\frac{\sin na}{a}\right)^2$$

$$+2 \int_0^a \frac{\Phi(z)}{z} \cdot \left(\frac{\sin nz}{z}\right)^2 dz - n \int_0^a \frac{\Phi(z)}{z} \cdot \frac{\sin 2nz}{z} dz.$$

Therefore

$$\lim_{n=\infty} \frac{1}{2n\pi} \int_0^a \Phi(z) \left(\frac{\sin nz}{z}\right)^2 dz = \lim_{n=\infty} \frac{1}{2n\pi} \Phi(a) \left(\frac{\sin na}{a}\right)^2$$

$$+2 \lim_{n=\infty} \frac{1}{2n\pi} \int_0^a \frac{\Phi(z)}{z} \left(\frac{\sin nz}{z}\right)^2 dz$$

$$-\lim_{n=\infty} \frac{1}{2\pi} \int_0^a \frac{\Phi(z)}{z} \frac{\sin 2nz}{z} dz. \quad (T_5).$$

Now, in the neighbourhood of $z=0$, in the present case

$$\Phi(z) = \int_0^z 2t \sin \frac{1}{t} dt - z^2 \sin \frac{1}{z}, \text{ } \textit{assuming} \text{ S to be zero.}$$

Therefore $\frac{\Phi(z)}{z}$ is a continuous function of z with 0 as its limit at $z=0$. Hence the first two parts of the right side in (T_5) are both zero. We have only to prove that

$$\operatorname*{Lim}_{n=\infty} \frac{1}{2\pi}\int_0^a \frac{\Phi(z)}{z} \frac{\sin 2nz}{z}\, dz=0.$$

But this follows from Riemann's theorem (T_1); because $\frac{\Phi(z)}{z^2}$ is integrable, being always numerically less than a finite quantity, say 2.

Therefore S_n tends to 0 as n tends to infinity; also the assumption made about S being 0 is justified, since

$$S=\operatorname*{Lim}_{z=0} \frac{1}{2z}\int_0^z \cos \frac{1}{t}\, dt$$

and this limit is 0.

§ 20.

37. The most comprehensive treatment for finding sufficient conditions for the summability (C1) of the Fourier series of a function at a point where it has a discontinuity of the second kind is that very recently given by G. Prasad.* He has proved the following criteria for the case in which, in the neighbourhood of the point of discontinuity x_0, $f(x_0+2t)+f(x_0-2t)=\chi(t)\cos\psi(t)$, χ and ψ being monotone and ψ being unbounded:

* "On the summability (C1) of the Fourier series of a function at a point where the function has an infinite discontinuity of the second kind" (*Bulletin of the Calcutta Math. Society*, Vol. 19, pp. 51-58).

XI. The Fourier series of $f(x)$ is summable (C1) at x_0, when

$$\psi \succ \log \frac{1}{t^2} \quad and \quad \frac{\chi}{t\psi'} \precsim 1;$$

and it is not summable (C1) at x_0, when $\psi \precsim \log \frac{1}{t^2}$.

XII. If $\psi \succ \log \frac{1}{t^2}$ and $\frac{\chi}{t\psi'} \succ 1$, the Fourier series of $f(x)$ is summable (C1) at x_0, if

$$\frac{\chi}{(t\psi')^2} \precsim 1 \quad and \quad \frac{\chi}{t\psi'} \prec t\sqrt{\psi''};$$

the series is not summable if

$$\frac{\chi}{(t\psi')^2} \precsim 1 \text{ and } \frac{\chi}{t\psi'} \succ t\sqrt{\psi''}.$$

XIII. If $\psi \succ \log \frac{1}{t^2}$ and $\frac{\chi}{(t\psi')^2} \succ 1$ but $\frac{\chi}{(t\psi')^3} \precsim 1$, then the Fourier series of $f(x)$ is not summable (C1) at x_0 when $\frac{\chi}{(t\psi')^2} \prec t\sqrt{\psi''}$.

To give you an idea of the method used for establishing these criteria, I will prove XI.

Proof:

(*a*) *Assume* that $S=0$; then note that in (T_5) of the preceding article

$$\Phi(z) \text{ behaves as } \frac{\chi}{\psi'} \sin \psi,$$

when

$$\psi \asymp \log \frac{1}{t^2} \text{ and } \frac{\chi}{t\psi'} \asymp 1.$$

For, integrating by parts,

$$\int_0^t \phi dt = \frac{\chi}{\psi'} \sin \psi - \int_0^t \frac{d}{dt}\left(\frac{\chi}{\psi'}\right) \sin \psi dt.$$

As $\frac{\chi}{\psi'} \asymp t$, $\frac{d}{dt}\frac{\chi}{\psi'} \asymp 1$.

Therefore Φ behaves as $\frac{\chi}{\psi'} \sin \psi + \frac{\cos \psi}{\psi'} - \frac{d}{dt}\left(\frac{\chi}{\psi'}\right)$

i. e. as $\frac{\chi}{\psi'} \sin \psi$.

Now the first limit in the right side of (T_5) is obviously 0, the second is 0 because $\frac{\Phi(z)}{z}$ tends to zero with z and the third limit tends to 0, because, obviously,

$$\frac{\chi}{\psi'} \asymp t\sqrt{\psi''}$$

and therefore Du Bois-Reymond's condition * for the vanishing of the limit is satisfied. Thus $\underset{n=\infty}{\text{Lim}}\, S_n = 0$.

The assumption regarding S is justified; for,

$$\underset{z=0}{\text{Lim}} \frac{1}{2z} \int_0^z \phi(t)dt = \underset{z=0}{\text{Lim}} \frac{\Phi(z)}{2z} = 0.$$

The case in which $\frac{\chi}{\psi'}$ is 1 is easily dealt with.

* *L. c.*, p. 37.

(*b*) If

$$\psi \asymp \log \frac{1}{t^2},$$

it is known that $\frac{1}{2n\pi}\int_0^z \chi \cos\psi \left(\frac{\sin nt}{t}\right)^2 dt$ behaves as $\chi\left(\frac{1}{n}\right)\cos\psi\left(\frac{1}{n}\right)$

which does not converge to any limit with increasing n.

§ 21.

38. The following are among the important properties of the Fejér's mean $S_n(x)$ of the Fourier series of a function $f(x)$:

(*i*) $\underset{n=\infty}{\text{Lim}}\ S_n$ exists at every point of continuity or ordinary discontinuity of $f(x)$, the limit being

$$\tfrac{1}{2}\{f(x+0)+f(x-0).$$

(*ii*) If M and m be the upper and lower limits of $f(x)$ in the interval $(-\pi, \pi)$, then S_n lies between M and m for values of x in that interval for every value of n.

(*iii*) $$\overline{\underset{n=\infty}{\text{Lim}}}\, S_n(x) \leq \tfrac{1}{2}\overline{F_1(+0)} > \underset{n=\infty}{\underline{\text{Lim}}}\ S_n(x) \geq \tfrac{1}{2}\underline{F_1(+0)},$$

where

$$F_1(z) \text{ denotes } f(x+2z)+f(x-2z).$$

(*iv*) If, excepting a finite number of points a_r where $f(x)$ may have ordinary discontinuity, $f(x)$ is everywhere continuous and has a continuous differential co-efficient $f'(x)$; then the derived series of the Fourier series of $f(x)$ is generally divergent for every x, whilst the Fejér mean of the derived series tends to $f'(x)$ at every point with the exception of the points a_r.

(*v*) Excepting possibly the points of a set of measure 0, $\underset{n=\infty}{\text{Lim}}\ S_n$ exists everywhere and equals $f(x)$.

The first property follows from Fejér's theorem (T_3) and Lebesgue's criterion.

The second property follows from the application of the first mean value theorem to the expression for S_n given in (a) of Art. 33. Similarly for the third property.

The fourth proper y* may be established as follows :—

Since $f'(x)$ is continuous except at the finite number of points a_r, the Fourier series of $f'(x)$, *viz.*,

$$\tfrac{1}{2}a_0' + \sum_{n=1}^{\infty} (a_n' \cos nx + b_n' \sin nx) \qquad \ldots \quad (1)$$

converges (C 1) to $f'(x)$ except at the points a_r. But, if there is only one point of discontinuity a, denoting $f(a-0)+f(a+0)$ by D_a and applying integration by parts, we find that

$$a_0' = \frac{D_a}{2\pi}, \qquad a_n' = \frac{D_a}{\pi} \cos na + nb_n,$$

$$b_n' = \frac{D_a}{\pi} \sin na - na_n, \quad n = 1, 2, 3, \ldots$$

Therefore (1) takes the form

$$\frac{D_a}{2\pi} + \sum_{1}^{\infty} \left\{ \frac{D_a}{\pi} \cos n(a-x) + (nb_n \cos nx - na_n \sin nx). \right\}$$

Now the part

$$\frac{D_a}{2\pi} + \sum_{1}^{\infty} \frac{D_a}{\pi} \cos n(a-x)$$

has its Fejér mean converging to 0 ; therefore it is proved that the Fejér mean of (1) is in the limit equal to the Fejér mean of the derived series of the Fourier series of $f(x)$.

The fifth property follows from Lebesgue's criterion and the integrability of $f(x)$.

* Fejér, *Math. Ann.*, Bd. 58, pp. 61-62.

§ 22.

39. In my first lecture I have already indicated the deduction of Jordan's criterion for the convergence of the Fourier series from the summability (C1) of that series to $\frac{1}{2}\{f(x+0)+f(x-0)\}$. I proceed to indicate how Dini's criterion can be deduced from the summability (C1) of the series.*

(*a*) As shown in (1) of Art. 33.

$$S_n - S = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} \phi(z)\left(\frac{\sin nz}{\sin z}\right)^2 dz,$$

$\phi(z)$ standing for $f(x+2z)+f(x-2z)-f(x+0)-f(x-0)$, S being

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

Hence

$$\frac{1}{p}\{nS_n-(n-p)S_{n-p}\}-S = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \phi(z)\frac{\sin^2 nz-\sin^2(n-p)z}{p\sin^2 z}\,dz$$

$$=\left(\frac{1}{\pi}\int_{\alpha}^{\frac{\pi}{2}} + \frac{1}{\pi}\int_0^{\alpha}\right)\phi(z)\frac{\sin^2 nz-\sin^2(n-p)z}{p\sin^2 z}dz,$$

where $\alpha>0$ is chosen to be arbitrarily small but independent of n and p.

The first integral in the above tends to 0 if p tends to infinity.

The modulus of the second integral is not greater than

$$\frac{1}{\pi}\int_0^{\alpha} |\,\phi(z)\,|\,\left|\frac{\sin(2n-p)z.\,\sin pz}{p\sin^2 z}\right| dz$$

$$<\frac{1}{\pi}\int_0^{\alpha} |\,\phi(z)\,|\,\frac{dz}{\sin z}$$

* Pollard, *Proc. L. M. S.*, Series 2, Vol. 10

since

$$|\sin(2n-p)z| \leq 1, \left|\frac{\sin pz}{\sin z}\right| < p.$$

Now,

if $\int_0^\pi |\phi(z)| \frac{dz}{z}$ is a convergent integral, so is

$$\frac{1}{\pi}\int_0^\alpha |\phi(z)| \frac{dz}{\sin z};$$

and this last integral can be made as small as we please by choice of α, because of the fact that $\phi(+0)=0$.

Thus if Dini's condition, *viz.*, that $\frac{|\phi(z)|}{z}$ is summable in the interval $(0, \alpha)$, is satisfied,

$$\operatorname{Lim} \frac{1}{p}\{nS_n-(n-p)S_{n-p}\}=S$$

however slowly p may tend to infinity as n tends to infinity.

(*b*) Now use the following theorem :

"If $a_m \longrightarrow 0$ with $\frac{1}{m}$, and we can find a sequence of integers p_n, such that p_n tends steadily to infinity with n, p_n/n tends steadily to 0, $p_n a_n$ tends to 0, and each of the means

$$\frac{nS_n-(n-p_n)Sn-p_n}{p_n}, \quad \frac{(n+p_n)Sn+p_n-nS_n}{p_n}$$

tends to S ; then the series Σu_n is convergent, and has the sum S." In this theorem the Fejér means are of Σu_n in which $u_n \longrightarrow 0$ as $n \longrightarrow \infty$, Σu_n contains an infinity of terms of either sign and is summable (C1) to S, and the a's are constructed as follows :

Let u_N be the negative term with the greatest modulus, u_{N_1} the negative term with the greatest modulus in $\sum_{n=N+1}^{\infty} u_n$, u_{N_2} that in $\sum_{n=N_1+1}^{\infty} u_n$, and so on.

Then N_1, N_2, ... certainly exist, because $u_n \longrightarrow 0$ as $n \longrightarrow \infty$. Take $a_n = -u_N$ $(n \leq N)$, $a_n = -u_{N_1}$ $(N < n \leq N_1)$, and so on. Then the a_n's, form a decreasing sequence of positive terms so that $u_n \geq -a_n$ for all values of n.

Now, since $a_n \longrightarrow 0$ we can certainly find a value of p, *viz.*, p_n so that all the conditions of the theorem are satisfied. Hence it is proved that $\sum u_n$ is convergent if Dini's condition is satisfied.

§ 23.

40. (*a*) From (T_5) of Art. 36 it follows that, neglecting those terms which vanish with $\frac{1}{n}$,

$$\sigma \equiv S_n(x) - S = \frac{2}{n\pi} \int_0^a \frac{\Phi(t)}{t} \left(\frac{\sin nt}{t}\right)^2 dt$$

$$- \frac{1}{\pi} \int_0^a \frac{\Phi(t)}{t} \frac{\sin 2nt}{t} dt \qquad \ldots (1)$$

Therefore, denoting by $\sum_n'$ and $\sum_n''$ respectively the parts of

$$\sum_n \equiv \frac{S_1 + S_2 + \ldots + S_n}{n} - S$$

due to the first and the second terms in the right side of (1), we have, by the first property of S_n given in Art. 38,

$$\lim_{n=\infty} \sum_n' = \lim_{t=0} \frac{\Phi(t)}{t} = \Phi'(0)$$

if $\Phi'(0)$ exists.

Therefore, if $\Phi'(0)$ is zero, $\sum_n$ behaves with increasing n as $\sum_n''$. But this equals

$$-\frac{1}{n\pi}\int_0^a \frac{\Phi(t)}{t}.\frac{\sin 2t+\ldots+\sin 2nt}{t}\,dt,$$

$$i.e.,\ -\frac{1}{n\pi}\int_0^a \frac{\Phi(t)}{t}.\frac{\sin (n+1)t \sin nt}{t \sin t}dt.$$

Therefore $\sum_n''$, and consequently $\sum_n$, equals, in the limit,

$$-\frac{1}{n\pi}\int_0^a \frac{\Phi(t)}{t}\left(\frac{\sin nt}{t}\right)^2$$

$$i.e.,\ \lim_{n=\infty} \sum_n = -\tfrac{1}{2}\Phi'(0).$$

Thus, if $\Phi'(0)=0$, $\lim_{n=\infty} \sum_n = 0$ so that the series is summable (C2) to S which is

$$\lim_{z=0} \frac{1}{2z}\int_0^z \{f(x+2t)+f(x-2t)\}dt.$$

The above condition, *viz.*, that $\Phi'(0)=0$, is Lebesgue's criterion * for summability (C2). XIV

* *Math. Ann.*, Bd. 61, pp. 278-279.

(*b*) I proceed to give Prasad's criticism * of Lebesgue's criterion XIV. He has shown that, when

$$f(x_0+2t)+f(x_0-2t)=\frac{d}{dt}\{t\chi(t)\cos\psi(t)\},$$

χ and ψ being both monotone and unbounded in the neighbourhood of $t=0$, and $\dfrac{\chi}{\psi'} \prec t$, the series is summable (C2) to 0 at x_0, although $\Phi'(0)$ is *not* zero.

Examples :

(*i*) For $f(x)=\dfrac{d}{dx}\left\{x\left(\log\dfrac{1}{x^2}\right)^{\frac{1}{3}}\cos\left(\log\dfrac{1}{x^2}\right)^{\frac{3}{2}}\right\}$, the series

is summable (C2) to 0 at $x=0$ although Lebesgue's criterion is not satisfied.

(*ii*) Similarly, for $f(x)=\dfrac{d}{dx}\left\{x\left(\log\dfrac{1}{x^2}\right)^{m}\cos\left(\log\dfrac{1}{x^2}\right)^{1+k}\right\}$,

$m>0$, $k>m$.

(*iii*) Also for $f(x)=\dfrac{d}{dx}\left\{x^{1-m}\cos\left(\dfrac{1}{x^k}\right)\right\}$, $\quad m>0, \quad k>m$,

$m+k<1$.

§ 24.

41. I will conclude this lecture by proving a number of results relating to summability (Ck), beginning with the following important theorem (T_6) of Hardy and Littlewood : †

" The Fourier series of a summable function $f(x)$, bounded in the neighbourhood of the particular value of x considered, is either summable (Ck) for *all* positive values of k (however small) or

* " On the failure of Lebesgue's criterion for the summability (C2) of the Fourier series of a function at a point where the function has a certain type of discontinuity of the second kind." (*Bulletin of the Calcutta Mathematical Society*, Vol. 19, pp. 25-28.)

† *Proc. L.M.S.*, Series 2, Vol. 17, pp. XIII-XV.

summable for *no* value of k (however large). The necessary and sufficient condition for summability is that

$$\frac{1}{2h}\int_{x-h}^{x+h} f(t)dt \longrightarrow S \qquad \ldots \quad (1)$$

when $h \longrightarrow 0$; and S is the sum of the series."

Proof:

(*a*) The first part of the theorem follows from the known theorems: * (*i*) If $f(x)$ is continuous then the series is summable (Ck) for every $k > 0$; (*ii*) similarly, if $f(x)$ is bounded the series is bounded (Ck) for every $k > 0$; (*iii*) if a series is bounded (Cr) and summable (Cs) where $r < s$, then it is summable (Ck) for every value of k between r and s.

(*b*) In order to prove that (1) is necessary and sufficient for summability, we may note at the outset that (1) is *sufficient* as it is sufficient for summability (C2) according to Lebesgue's criterion XIV given in the preceding article. To prove that (1) is *necessary*, the property of the bound edness of the function is utilized as follows:

According to Fejér,† if the series is summable (C1) to S then

$$\frac{\phi(h)}{16h^4} \equiv \frac{\Delta^4_{2h} F_4(x)}{16h^4}$$

$$\equiv \frac{F_4(x+4h) - 4F_4(x+2h) + 6F_4(x) - 4F_4(x-2h) + F_4(x-4h)}{16h^4} \longrightarrow S,$$

when $h \longrightarrow 0$, $F_4(x)$ being the 4th iterated integral of $f(x)$. Taking, as we may do without loss of generality, $S = 0$, we have

$$\phi(h) = o(h^4),$$

$$\text{Also } \phi'''(h) = 64\left\{\chi(h) - \frac{1}{2}\chi\left(\frac{1}{2}h\right)\right\}, \text{ where}$$

$$\chi(h) \equiv F_1(x+4h) - F_1(x-4h)$$

is the integral of a bounded function. From this it is easily deduced ‡ that $\phi'''(h) = o(h)$ and hence that $\chi(h) = o(h)$; which completes the proof the theorem (T_6).

* See *Proc. L.M.S.*, Series 2, Vol. 11, p. 411.

† *Math. Ann.*, Bd. 58, p. 69.

‡ See Hardy and Littlewood, l.c., p. XV.

42. As corollaries of the general theorem of Knopp quoted in the preceding, we have (1), by applying the first property of S_n given in Art. 38, the result of M. Riesz, Chapman and W. H. Young, *viz.*, that for $k > 0$ the Fourier series is summable (Ck) at every point of continuity or ordinary discontinuity of $f(x)$ to

$$\frac{1}{2}\left\{ f(x+0)+f(x-0) \right\};$$

and (2), by applying the fourth property, Hardy's extension, *viz.*, that the series is summable (Ck) to $f(x)$ for every value of x with the possible exception of those values which form a set of measure 0.

43. I proceed to prove the following theorem * of Hardy and Littlewood, already stated in the first lecture: The necessary and sufficient condition that the Fourier series corresponding to $f(x)$ should be summable (Ck) to S, for *some value or another* of k, at the point x_0, is that there should be an integer r such that if

$$\phi(t)=f(x_0+t)+f(x_0-t)-2S,$$

$$\lim_{t=0} \phi_r(t)=0, \qquad \ldots \qquad (T_7)$$

where

$$\phi_1(t)=\frac{1}{t}\int_0^t \phi(t_1)dt_1,\ \phi_2(t)=\frac{1}{t}\int_0^t \phi_1(t_1)dt_1,\ \phi_3(t)=\frac{1}{t}\int_0^t \phi_2(t_1)dt_1,$$

$$\ldots \qquad \ldots \qquad \ldots$$

Proof:

(*a*) *Preliminary assumptions and definitions.*

(*i*) The Fourier series of $f(t)$ for $t=x_0$ is the same as that of $\phi(t)$ for $t=0$, leaving aside the constant term and a factor $\frac{1}{2}$. We may, therefore, confine our attention to even functions, and to the special value $t=0$; and we may suppose that $S=0$. We may also suppose that $a_0=o$, so that

$$\phi(t) \backsim a_1 \cos t + a_2 \cos 2t + \ldots$$

* "Solution of the Cesàro summability problem for power series and Fourier series." (*Math. Zeit.*, Bd. 19, 1923, pp. 67-96.)

(*ii*) Let us introduce certain auxiliary functions, *viz.*,

$$\psi_0=\phi_0=\phi\,;\psi_1=\frac{1}{2}\cot\frac{1}{2}t\,.\int_0^t\psi_0(t_1)dt_1+\gamma_1\sin^2\frac{1}{2}t,$$

$$\psi_2=\frac{1}{2}\left[\cot\frac{1}{2}t\int_0^t\psi_1(t_1)dt_1+\gamma_2\sin^2\frac{1}{2}t,\ldots\ldots\right.$$

where the γ's are constants so that

$$\int_{-\pi}^{\pi}\psi_1(t)dt=0,\int_{-\pi}^{\pi}\psi_2(t)dt=0,\ldots\int_{-\pi}^{\pi}\psi_s(t)dt\equiv 0,\ldots$$

and we may write

$$\psi_s(t)\backsim a_1{}^s\cos t+a_2{}^s\cos 2t+\ldots$$

(*iii*) If ψ_r exists so does the corresponding ϕ_r, and their difference is $o(t)$. For, obviously this is 0 for $r=0$; and if it is true for $r=m$, then it is true for $r=m+1$, because

$$\psi_{m+1}=\left(\frac{1}{t}+O(t)\right)\int_0^t\{\phi_m(t_1)+0(t_1)\}dt_1+0(t)=\phi_{m+1}+0(t).$$

(iv). The following basic theorem in the pure theory of infinite series holds: The necessary and sufficient condition that $A\equiv\Sigma a_n$ should be summable (C, r) is that there should be a system of numbers $a_{n,s}$ ($s=0, 1, 2,\ldots r+1$; $n=0, 1, 2,\ldots$) such that

$$a_{0,s}=a_0=0,\ a_{n,0}=a_n(n>0),$$

$$a_{n,s-1}=n(a_{n,s}-a_{n+1,s}),\ n>0,\ s>0,$$

and $A_{r+1}\equiv\Sigma a_{n,r+1}$ is summable (C,-1).

In these circumstances

$$a_{n,s}=\sum_n^\infty\frac{a_{\nu,s-1}}{\nu}\ (C,r-s),\text{ for } s=1, 2,\ldots,r+1;$$

the series A_s * is summable (C, $r-s$) and the sums of all the series are the same.

(*b*) *Case of integrable* $(\phi)^2$.†

$$\sum_n^N \frac{a_\nu}{\nu} = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(t)\sum_n^N \frac{\cos \nu t}{\nu}\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\Phi_1(t)\sum_n^N \sin \nu t\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\left(n-\frac{1}{2}\right)t - \cos\left(N+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,\Phi_1(t)\,dt$$

* $A \equiv \Sigma a_n$ and A is used to denote the series or its sum when the series is summable. We write $A_n = A_n^0 = a_0 + a_1 + \ldots a_n$, $A_n^1 = A_0^0 + A_1^0 + \ldots + A_n^0$, $A_n^2 = A_0^1 + \ldots + A_n^1$, ... with a corresponding notation in other letters. Summability (C, 0) is the same thing as ordinary convergence; and A is said to be summable (C, -1) if it is convergent and $n\,a_n \to 0$.

If $C_n = 1$. so that

$$C^r{}_n = \frac{(n+r)(n+r-1)\ldots(n+1)}{r!} = \frac{\Gamma(n+r+1)}{\Gamma(n-1)\Gamma(r+1)}$$

and $A_\sigma^r \backsim C^r{}_n A$, the series is said to be summable (C, r) to sum A and we write

$$\Sigma a_n = A(C,r).$$

† It is easily seen that if ϕ^2 is integrable, then $(\phi_r)^2$ and $(\psi_r)^2$ are integrable. It is obviously sufficient for proving the above statement, to prove that $(\phi_1)^2$ is integrable, *i.e.*, that

$$\int_\epsilon^\delta (\phi_1)^2 dt \text{ is bounded when } \epsilon \to 0. \quad \text{But, if}$$

$$\Phi_1(t) = \int_0^t \phi(t_1)dt_1, \text{ we have } (\Phi_1')^2 \leq t\int_0^t \phi^2 dt_1 = O(t)$$

$$\text{and so } \int_\epsilon^\delta (\phi_1)^2 dt = \frac{\{\Phi_1(\epsilon)\}^2}{\epsilon} - \frac{\{\Phi_1(\delta)\}^2}{\delta} + 2\int_\epsilon^\delta \phi_1 \phi\, dt = O(1) + O(1) + O\left(\sqrt{\int_\epsilon^\delta \phi_1{}^2 dt}\right)$$

which is impossible unless the left side is bounded.

which tends to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_1(t)\cot\frac{1}{2}\ t\ .\quad \cos nt\ dt+\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_1(t)\sin nt\ dt$$

when N tends to ∞, since $\Phi_1(t)\cot\frac{1}{2}\ t$ is integrable.

Hence, in the notation of the above basic theorem,

$$a_{n,1}=a_n^1+b_{n,1}=a_n^1+b'_{n,1}+b''_{n,1}\ ,\ \text{where}$$

$$b'_{n,1}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_1(t)\sin nt\ dt,\ b''_{n,1}=-\frac{\gamma_1}{2\pi}\int_{-\pi}^{\pi}\sin^2\frac{1}{2}\ t\cos nt\ dt,$$

so that $b''_{n,1}=\frac{1}{2}\gamma_1$ if $n=1$ and $b''_{n,1}=0$ if $n>1$.

The series $\sum b''_{n,1}$ is convergent to $\frac{1}{2}\ \gamma_1$. Also

$$\sum_n^N b'_{n,1}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\frac{1}{2}\ t-\cos\left(N+\frac{1}{2}\right)t}{2\sin\frac{1}{2}\ t}\ \Phi_1\ dt$$

$$\longrightarrow\frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{2}\cot\frac{1}{2}\ t\ \Phi_1(t)dt,$$

when $N\longrightarrow\infty$. Hence $\sum b_{n,1}$ is convergent and its sum is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\cot\frac{1}{2}\ t\ \Phi_1(t)\ dt+\frac{1}{2}\gamma_1=\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi_1(t)dt=0.$$

Also $b'_{n,1}$ being the Fourier sine coefficent of an integral, is $0\left(\frac{1}{n}\right)$, and therefore $b_{n,1}$ is of the same form. Thus $\sum b_{n,1}$ is summable $(C,-1)$

to zero, and it is proved that $a_{n,1}$ differs from a_n^1 by the general term of a series B_1, summable $(C,-1)$ to 0.

As ψ_1 posesses all the relevant properties of ψ_0, we may repeat the argument.

The series $B_2, B_3, \ldots$ formed from B_1 as $A_2, A_3, \ldots$ are formed from A_1 are all summable $(C,-1)$ to 0, and so are all other series of similar type introduced in the argument. We thus obtain the following result: If ϕ^2 is integrable, then $a_{n,s}$ differs from the Fourier coefficient of ψ_s by the general term of a series summable $(C,-1)$ to 1. ... (1)

Also it has been shown by Fatou * that: if $a_n = 0\left(\frac{1}{n}\right)$, then the necessary and sufficient condition for the convergence of the series Σa_n is that ϕ_1 should tend to that series. ... (2)

(*c*) (*i*) The condition of the theorem is *necessary*. For, if A is summable (C, r) to zero, A_{r+1} is summable $(C,-1)$ to 0 by the basic theorem given under (iv) of (*a*) above. Also by (1), A^{r+1} is summable $(C,-1)$ to 0. Hence it follows by the application of (2), that ψ_{r+1} or ϕ_{r+1} tends to 0.

(*ii*) The condition is *sufficient*. For, if ϕ_r or ψ_r tends to 0, A^r is summable $(C, 1)$ to 0 by Fejér's theorem. Also A_r is summable $(C, 1)$ to 0 by (1). Therefore A is summable $(C, r+1)$ to 0 by the basic theorem quoted above.

(*d*) The integrability of ϕ does not necessarily imply the integrability of ϕ_1. If ϕ^2 is not integrable, the proof of the theorem will be on the same lines as in the case in which ϕ^2 is integrable, but more delicate considerations † are required.

* *Acta Math.*, t. 30 (1906), pp. 335-400.

† See pp. 88-95 of the paper of Hardy and Littlewood quoted above.

FIFTH LECTURE.

Strong Summability.

§ 25.

43. As stated in the first lecture, the notion of "strong summability" was introduced by Hardy and Littlewood in 1913; and indeed the following are the words used by them in introducing that notion *: "Let $f(u)$ be a function of u, summable and possessing the period 2π, and let us investigate an ordinary value s of u, that is to say, a value of u for which the expression

$$s=\tfrac{1}{2}[f(x+0)+f(x-0)]$$

has a determinate value. One knows from the known theorem of Fejér that the series of Fourier

$$\tfrac{1}{2}A_0+\Sigma A_m=\tfrac{1}{2}a_0+\Sigma(a_m\cos mu+b_m\sin mu)$$

of $f(u)$ is summable for $u=x$ by the mean of Cesàro of the first order and to the sum s. It is this which we will express by writing

$$\operatorname{Lim}\frac{(s_0-s)+(s_1-s)+\ldots+(s_n-s)}{n+1}=0,$$

where

$$s_m=\tfrac{1}{2}A_0+A_1+\ldots+A_m.$$

One owes to Messrs. Lebesgue, Marcel Riesz and Chapman very interesting generalizations of the theorem of Mr. Fejér. We have found, for the function of summable square, a result of a slightly different character. The square of $f(u)$ being summable, one has for $u=x$,

* "Sur la série de Fourier d'une fonction à carre sommable" (*Comptes rendus*, t. 156, pp. 1307-1309).

$$\operatorname{Lim} \frac{(s_0-s)^2+(s_1-s)^2+\ldots+(s_n-s)^2}{n+1}=0, \qquad (T_s)$$

and

$$\operatorname{Lim} \frac{|s_0-s|+|s_1-s|+\ldots+|s_n-s|}{n+1}=0.$$

The second equality is a corollary of the first, as one sees immediately by applying the inequality of Schwarz."*

44. I proceed to prove the aforesaid theorem, reproducing as far as practicable the original proof of Hardy and Littlewood on account of its historical interest.

Proof:

(*a*) First assume that $s=0$; also take $T_m=\frac{1}{2}A_0+A_1+\cdots+A_{m-1}+\frac{1}{2}A_m$, so that

$$s_m=T_m+\tfrac{1}{2}A_m.$$

Now

$$\frac{(s_0-s)^2+(s_1-s)^2+\ldots+(s_n-s)^2}{n+1}=\frac{s_0^2+s_1^2+\ldots+s_n^2}{n+1}$$

$$=\frac{T_0^2+T_1^2+\ldots+T_n^2}{n+1}+\tau_n,$$

where

$$\tau_n=\frac{\sum_0^n A_m T_m+\frac{1}{4}\sum_0^n A_m^2}{n+1},$$

which obviously tends to 0 with $\frac{1}{n}$.

Therefore it will be sufficient to prove that

$$\operatorname{Lim}_{n=\infty} \frac{T_0^2+T_1^2+\ldots+T_n^2}{n+1}=0.$$

* If (f_1, f_2) denote $\frac{1}{\pi}\int_{-\pi}^{\pi} f_1 f_2\, dx$, then $(f_1, f_2)^2 \leq (f_1, f_1), (f_2, f_2)$.

Now

$$T_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t).\ \frac{\sin(2n-1)\frac{t}{2}}{\sin\frac{t}{2}}\,dt$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cos nt$$

(by the result in Art. 17).

Thus

$$T_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t).\ \sin nt,\ \cot\frac{t}{2}\,dt.$$

Hence

$$2\sum_{n=1}^{\infty} T_n\, r^n \sin n\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} dt f(x+t)\ \cot\frac{t}{2}\left\{\sum_{1}^{\infty} r^n\cos n\ (\theta-t) - \sum_{1}^{\infty} r^n\cos n\ (\theta+t)\right\}$$

$r<1$.

But

$$\sum_{1}^{\infty} r^n\cos n\ (\theta-t) = \frac{r\{\cos(\theta-t)-r\}}{1-2r\cos(\theta-t)+r^2},$$

$$\sum_{1}^{\infty} r^n\cos n\ (\theta+t) = \frac{r\{\cos(\theta+t)-r\}}{1-2r\cos(\theta+t)+r^2}.$$

Therefore

$$2\sum_{1}^{\infty} T_n\, r^n \sin n\,\theta$$

$$=\frac{r}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}\cdot\left\{\frac{\cos(\theta-t)-r}{1-2r\cos(\theta-t)+r^2}\right.$$

$$\left.-\frac{\cos(\theta+t)-r}{1-2r\cos(\theta+t)+r^2}\right\}dt$$

$$= \frac{r(1-r^2)\sin\theta}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t)\sin t.\ \cot\frac{t}{2}dt}{\{1-2r\cos(\theta-t)+r^2\}\{1-2r\cos(\theta+t)+r^2\}}$$

$$= \frac{r(1-r^2)\sin\theta}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t).\ 2\cos^2\frac{t}{2}\,dt}{\{1-2r\cos(\theta-t)+r^2\}\{1-2r\cos(\theta+t)\}+r^2\}}$$

$$= \frac{r(1-r^2)\sin\theta}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t).(1+\cos t)dt}{A-2B\cos t+C\cos^2 t},$$

where

$$A=(1+r^4+2r^2\cos 2\theta),\ B=2r(1+r^2)\cos\theta,\ C=4r^2.$$

(*b*) From the above expression it can be deduced that, " corresponding to a quantity $\epsilon>0$ another quantity $\delta>0$ can always be found such that

$$\left| \sum_{1}^{\infty} T_n r^n \sin n\theta \right| < \epsilon\left\{ \sqrt{\frac{1}{1-r}} + \frac{|\sin\theta|}{1-2r^2\cos 2\theta+r^2} \right\},$$

$$\text{for} \qquad 1-\delta < r < 1.$$

Integrating from $\theta=-\pi$ to $\theta=\pi$ the above inequality after squaring both sides of it, we find

$$\lim_{r=1}\left\{(1-r)\sum_{1}^{\infty} T_n^2 r^{2n}\right\}=0$$

and a fortiori

$$\lim_{r=1}\left\{(1-r)\sum_{1}^{\nu} T_n^2 r^{2n}\right\}=0.$$

Hence the result

$$\lim_{\nu=\infty}\frac{1}{\nu}\sum_{1}^{\nu} T_n^2=0$$

follows if we take $r=1-\frac{1}{\nu}$."

§ 26.

45. With the same suppositions as those of Hardy and Littlewood, Fejér has given the following ingenious and simple proof * of their theorem:

Let z denote a complex variable $r\,e^{i\theta}$ and let $g(z)$ denote the power series $\sum_0^\infty c_\gamma\, z^\gamma$ which is assumed to be convergent for $|\,z\,|<1$, the c's being the co-efficients of the Fourier series under consideration, *viz.*, $\sum_0^\infty c_\gamma \cos \gamma t$. Then, denoting $c_0+c_1+\ldots+c_\gamma$ by s_γ, we have

$$\frac{g(z)}{1-z}=\sum_0^\infty s_\gamma\, z^\gamma.$$

Hence, by Parseval's theorem for power series, when $0 \leq r < 1$,

$$\sum_0^\infty |\,s_\gamma\,|^2 r^{2\gamma}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\,\frac{g(z)}{1-z}\,\right|^2 d\theta$$

$$=\frac{1}{2\pi}\left\{\int_{-\pi}^{-\epsilon}+\int_{-\epsilon}^{\epsilon}+\int_{\epsilon}^{\pi}\left|\,\frac{g(z)}{1-z}\,\right|^2 d\theta\right\},\ \ldots\quad (1)$$

where, corresponding to an arbitrarily small but fixed quantity $\eta > 0$, ϵ and δ are positive quantities such that

$$|\,g(z)\,|=|\,g(r\,e^{i\theta})\,|<\eta,$$

for

$$1-\delta \leq r < 1,\quad -\epsilon \leq \theta \leq \epsilon.$$

Therefore the three parts of the last term of (1) are easily seen to be less than

$$\frac{G}{\sin^2\epsilon},\quad \frac{\eta^2}{1-r^2}\quad \text{and}\quad \frac{G}{\sin^2\epsilon}$$

* "Fourierreihe und Potenzreihe" (*Monatshefte für Mathematik und Physik*, Bd. XXVIII, 1917, pp. 64-76).

respectively, where

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\, g(z)\,|^2 d\theta < G.$$

Hence

$$(1-r^2)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{g(z)}{1-z}\right|^2 d\theta < 2\eta^2, \qquad \ldots \quad (2)$$

if δ is sufficiently small.

Therefore, writing ρ for r^2, we have from (1) and (2),

$$\lim_{\rho=1}\ (1-\rho)\sum_{\gamma=0}^{\infty} s_\gamma^2\,\rho^\gamma = 0.$$

Using Tauber's artifice, put $1-\frac{1}{n+1}$ for ρ; then the above becomes

$$\lim_{n=\infty}\frac{1}{n+1}\sum_{\gamma=0}^{n} s_\gamma^2 = 0.$$

§ 27.

46. I proceed to give the proof of the following generalization of the result of Hardy and Littlewood by Carleman: "If s_ν is the sum of the first $\nu+1$ terms of the Fourier series of a summable function $f(x)$, and for a particular value of x,

$$\int_0^\delta |\, f(x+t)+f(x-t)-2f(x)\,|\, dt = o(\delta),$$

$$\int_0^\delta \{f(x+t)+f(x-t)-2f(x)\}^2 dt = O(\delta),$$

when $\delta \longrightarrow 0$, then

$$\lim_{n\longrightarrow\infty}\frac{1}{n+1}\sum_{\nu=0}^{n} |\, s_\nu - f(x)\,|^k = 0$$

for every positive value of k."

Proof :

For the sake of simplicity, take $x=0$; also denote

$$f(t)+f(-t)-2f(0)$$

by $2\phi(t)$.

(*i*) Now it is easily seen that

$$s_\nu - f(0)=\frac{1}{\pi}\int_0^\pi \phi(t).\frac{\sin(\nu+\frac{1}{2})t}{\sin\frac{1}{2}t}dt=\alpha_\nu+\beta_\nu+\gamma_\nu,$$

where

$$\alpha_\nu=\frac{1}{\pi}\int_0^\delta \sin\nu t.\cot\tfrac{1}{2}t.\phi(t)dt,\quad \beta_\nu=\frac{1}{\pi}\int_\delta^\pi \sin\nu t\cot\tfrac{1}{2}t.\phi(t)dt,$$

$$\gamma_\nu=\frac{1}{\pi}\int_0^\pi \cos\nu t\,\phi(t)dt.$$

Also from Minkowski's inequality,* it follows that

$$\left\{\sum_{\nu=0}^{n} |s_\nu - f(0)|^{2p}\right\}^{\frac{1}{2p}} \le \left\{\sum_{\nu=0}^{n}|\alpha_\nu|^{2p}\right\}^{\frac{1}{2p}}$$

$$+\left\{\sum_{\nu=0}^{n}|\beta_\nu|^{2p}\right\}^{\frac{1}{2p}}+\left\{\sum_{\nu=0}^{n}|\gamma_\nu|^{2p}\right\}^{\frac{1}{2p}}.$$

* Minkowski's inequality in its classical form is this: If $a_\nu \ge 0$, $b_\nu \ge 0$, and $p \ge 1$; then

$$\left\{\sum_{\nu=1}^{n}(a_\nu+b_\nu)^p\right\}^{\frac{1}{p}} \le \left(\sum_{\nu=1}^{n}a_\nu^p\right)^{\frac{1}{p}}+\left(\sum_{\nu=1}^{n}b_\nu^p\right)^{\frac{1}{p}}.$$

So the proof is mainly concerned with the consideration of the sums

$$\sum_{\nu=0}^{n} |\, a_\nu \,|^{2p}, \quad \sum_{\nu=0}^{n} |\, \beta_\nu \,|^{2p}, \quad \sum_{\nu=0}^{n} |\, \gamma_\nu \,|^{2p}.$$

(*ii*) It can be shown that the first sum is $n^{1+\frac{1}{2p}}\, o(\delta)$ and that the third sum equals $O(1)$.

For, taking up the first sum first, for $0 < t < \delta$,

$$|\sin \nu t . \tfrac{1}{2} \cot \tfrac{1}{2} t| \leq \nu \{1 + o(1)\},$$

where the symbol o refers to $\delta \to 0$ and implies a constant independent of ν. Hence $|\, a_\nu \,| \leq \frac{2\nu}{\pi} \left\{ 1 + o(1) \right\} \int_0^\delta |\, \phi(t) \,|\, dt = \nu .\, o(\delta)$;

and, consequently,

$$\left\{ \sum_{\nu=0}^{n} |\, a_\nu \,|^{2p} \right\}^{\frac{1}{2p}} \leq \left\{ \sum_{\nu=0}^{n} \nu^{2p} \right\}^{\frac{1}{2p}} o(\delta) < \left(n^{2p+1} \right)^{\frac{1}{2p}} o(\delta),$$

i.e., the first sum $= n^{+\frac{1}{2p}}\, o(\delta)$.

Again,

$$\gamma_\nu = \frac{1}{\pi} \int_0^\pi \cos \nu t \, \phi(t) dt$$

so that

$$\phi(x) \backsim \gamma_0 + 2 \sum_{1}^{\infty} \gamma_\nu \cos \nu x ;$$

and, since ϕ is of summable square,

$$\gamma_0^{\,2} + 2 \sum_{1}^{\infty} \gamma_\nu^{\,2} = \frac{1}{\pi} \int_0^\pi \{\phi(t)\}^2 dt.$$

Hence

$$\sum_{\nu=0}^{\infty} \left| \gamma_\nu \right|^{2p} \leq \left\{ \sum_{\nu=0}^{\infty} \left| \gamma_\nu \right|^2 \right\}^p = O(1),$$

i.e., the third sum $= O(1)$

(*iii*) I proceed now to prove that the second sum

$$< K p \delta^{-\frac{1}{2p}},$$

where K is an absolute constant.

It may be noted at the outset that, as can be proved* without

* From $\phi_s(x) \backsim \sum_{\nu=1}^{\infty} c_\nu(s) \sin \nu x$, we have

$$\phi_s(x+y) \backsim \sum_{\nu=1}^{\infty} c_\nu(s) (\sin \nu x \cos \nu y + \cos \nu x \sin \nu y)$$

Hence

$$\phi_{2,s}(x) \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_s(y) \phi_s(x+y) dy = \sum_{\nu=1}^{\infty} \left\{ c_\nu(s) \right\}^2 \cos \nu x,$$

the series being absolutely convergent.

Similarly

$$\phi_{3,s}(x) \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_{2,s}(y) \phi_s(x+y) dy = \sum_{\nu=1}^{\infty} \left\{ c_\nu(s) \right\}^3 \sin \nu x, \text{ and generally}$$

$$\phi_{2p,s}(x) = \sum_{\nu=1}^{\infty} \left\{ c_\nu(s) \right\}^{2p} \cos \nu x. \quad \text{In particular, } \sum_{\nu=1}^{\infty} \left\{ c_\nu(s) \right\}^{2p} = \phi_{2p,s}(0).$$

$$\text{Now } \left| \phi_{2,s}(x) \right| \leq \frac{1}{\pi} \left[\int_{-\pi}^{\pi} \left\{ \phi_s(y) \right\}^2 dy \int_{-\pi}^{\pi} \{\phi_s(x+y)\}^2 dy \right]^{\frac{1}{2}}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \phi_s(y) \right|^2 dy = \frac{2}{\pi} \theta(s).$$

difficulty,

$$\sum_{\nu=1}^{\infty}\left\{c_\nu(s)\right\}^{2p}\leq\left(\frac{2}{\pi}\right)^{2p-1}\theta(s)\{\rho(s)\}^{2p-2},$$

where $c_\nu(s)$ is the typical Fourier co-efficient of the function $\phi_s(x)$ which is odd, equal to 0 for $s<x<\pi$ and equal to $\phi(x)$ for $0<x<s$,

$$\int_0^s|\phi(t)|\,dt=\rho(s),\quad\int_0^s|\phi(t)|^2dt=\theta(s).$$

Now

$$\beta_\nu=\frac{2}{\pi}\int_\delta^\pi \text{sin. } \nu t\,\frac{1}{2}\,\text{cot.}\frac{t}{2}\,.\,\phi(t)dt$$

$$=-\frac{1}{2}\,\cot\frac{\delta}{2}\,.\,c_\nu(\delta)+\frac{1}{4}\int_\delta^\pi c_\nu(t)\,\text{cosec}^2\frac{t}{2}\,dt$$

on integrating by parts.

Hence $|\phi_{3,1}(x)|\leq\frac{1}{\pi}\int_{-\pi}^{\pi}|\phi_{2,1}(y)|\,|\phi_1(x+y)|\,dy\leq\frac{2}{\pi^2}\theta(s)\int_{-\pi}^{\pi}|\phi_1(x+y)|\,dy$

$$=\left(\frac{2}{\pi}\right)^3\theta(s)\rho(s).$$

Proceeding in this way, we get generally $|\phi_{2p,1}|\leq\left(\frac{2}{\pi}\right)^{2p-1}\theta(s)\{\rho(s)\}^{2p-2}$,

and putting $x=0$, $\sum_{\nu=1}^{\infty}\left\{c_\nu(s)\right\}^{2p}\leq\left(\frac{2}{\pi}\right)^{2p-1}\theta(s)\left\{\rho(s)\right\}^{2p-2}$.

So

$$\left| \sum_{\nu=1}^{\infty} \xi_\nu \beta_\nu \right| \leq \frac{1}{2} \cot \frac{\delta}{2} \sum_{\nu=1}^{\infty} \left| \xi_\nu c_\nu (\delta) \right|$$

$$+\frac{1}{4} \int_{\delta}^{\pi} \operatorname{cosec}^2 \frac{1}{2} t \sum_{\nu=1}^{\infty} \left| \xi_\nu c_\nu (t) \right| dt$$

for any values of the ξ's for which the right side exists.

Suppose

$$\sum_{\nu=1}^{\infty} \left| \xi_\nu \right|^{2p/2p-1} = 1. \quad \text{Then} \quad \operatorname*{Max}_{\xi} \left| \sum_{\nu=1}^{\infty} a_\nu \xi_\nu \right|$$

$$= \left\{ \sum_{\nu=1}^{\infty} \left| a_\nu \right|^{2p} \right\}^{\frac{1}{2p}}$$

if the right side is convergent; and this is so if $a_\nu = c_\nu (t)$.

Hence

$$\left\{ \sum_{\nu=1}^{\infty} \left| \beta_\nu \right|^{2p} \right\}^{\frac{1}{2p}} \leq \tfrac{1}{2} \cot \frac{\delta}{2} \left\{ \sum_{\nu=1}^{\infty} \left| c_\nu (\delta) \right|^{2p} \right\}^{\frac{1}{2p}}$$

$$+\tfrac{1}{4} \int_{\delta}^{\pi} \operatorname{cosec}^2 \frac{t}{2} \left\{ \sum_{1}^{\infty} \left| c_\nu (t) \right|^{2p} \right\}^{\frac{1}{2p}} dt.$$

Now, postulating that each K denotes an absolute constant, we know that

$$\rho(s) < K_1 s, \; \theta(s) < K_2 s.$$

Hence

$$\sum_{\nu=1}^{\infty}\left\{\ c_\nu\ (s)\ \right\}^{2p}\leq\left(\frac{2}{\pi}\right)^{2p-1}\mathrm{K}_2 s\ \left\{\mathrm{K}_1 s\ \right\}^{2p-2},\ \text{and so}$$

$$\left[\ \sum_{\nu=1}^{\infty}\left\{\ c_\nu\ (s)\right\}^{2p}\right]^{\frac{1}{2p}}<\mathrm{K}_3 s^{\frac{2p-1}{2p}}.$$

Thus

$$\left\{\ \sum_{\nu=1}^{\infty}\left|\ \beta_\nu\ \right|^{2p}\right\}^{\frac{1}{2p}}\leq\frac{\mathrm{K}'}{\delta}\ \delta^{\frac{2p-1}{2p}}+\mathrm{K}'\int_{\delta}^{\pi}t^{-\frac{1}{2p-1}}\,dt$$

$$<\mathrm{K}'\,\delta^{-\frac{1}{2p}}+\mathrm{K}'.\ \ 2p\,\delta^{-\frac{1}{2p}}<\mathrm{K}\,p\ \delta^{-\frac{1}{2p}}.$$

(*iv*) Combining the inequalities obtained under (*ii*) and (*iii*), we have

$$\left\{\ \sum_{\nu=0}^{n}\left|\ s_\nu\ -f\,(0)\ \right|^{2p}\right\}^{\frac{1}{2p}}<n^{1+\frac{1}{2p}}\ o(\delta)+\mathrm{K}p\delta^{-\frac{1}{2p}}+\mathrm{O}\ (1),$$

i.e.,$$\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}\left|\ s_\nu\ -f\,(0)\right|^{2p}\right\}^{\frac{1}{2p}}<n\delta\epsilon\,(\delta)+\frac{\mathrm{K}p}{(n\delta)^{\frac{1}{2p}}}+\frac{\mathrm{K}}{n^{\frac{1}{2p}}},$$

where $\epsilon\ (\delta)=o\ (1)$ when $\delta\longrightarrow 0$.

(*v*) Take now $\delta=\frac{b}{n}$ where b is large enough to make $\mathrm{K}pb^{-\frac{1}{2p}}<\frac{1}{2}\epsilon$

and then assume n to be large enough to make

$$b\epsilon(\delta)+\mathrm{K}n^{-\frac{1}{2p}}<\tfrac{1}{2}\epsilon,$$

where

$$\epsilon(\delta)=\epsilon\left(\frac{b}{n}\right)\longrightarrow 0\ \text{when}\ n\longrightarrow\infty,$$

Hence

$$\overline{\lim_{n=\infty}} \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} \left| s_\nu - f(0) \right|^{q\,p} \right\} < \epsilon^{q\,p},$$

and so, since ϵ is as small as we please, it is proved that

$$\lim_{n=\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \left| s_\nu - f(0) \right|^{q\,p} = 0.$$

§ 28.

47. By using Hausdorff's completed form of Parseval's theorem, Sutton gave a simpler proof of Carleman's theorem which he also generalized to the form:

"If, with the notation

$$2\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$\int_0^\delta |\phi(t)|^q dt = o(\delta), \quad (0<q<1),$$

and

$$\int_0^\delta |\phi(t)|^p dt = O(\delta), \quad (1<p<2),$$

then

$$\lim_{n=\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \left| s_\nu - f(0) \right|^{p/p-1} = 0"$$

Sutton also proved that if we replace the first condition in the above by the more general condition

$$\int_0^\delta \phi(t)\, dt = o(\delta),$$

then

$$\operatorname*{Lim}_{n=\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \left| \sigma_\nu - f(x) \right|^{p/p-1} = o,$$

where σ_ν denotes the arithmetic mean of the first ν terms of the Fourier series of $f(x)$.

§ 29.

48. The latest published investigation of the subject of strong summability is by Hardy and Littlewood who give the following theorem which, according to them, has, "more the air of finality" than the theorem of Carleman and Sutton's generalization of it.

If $p>1$ and

$$\int_0^t |\phi(u)|^p du = O(t), \quad \int_0^t \phi(u)\, du = o(t),$$

$\phi(t)$ standing for $\frac{1}{2}\{f(x+t)+f(x-t)-2s\}$;

then

$$\sum_{m=0}^{n} |s_m - s|^q = o(n)$$

for every positive q.

Proof:

(a) For the sake of simplicity, take $f(t)$ to be an even function with zero mean value so that

$$f(t) \backsim \sum_{1}^{\infty} a_n \cos nt;$$

also take $x=0$, $s=0$. Then $\phi(t)=f(t)$, $s_m=a_1+a_2+\ldots+a_m$.

If the theorem is proved in this case it will be true generally.

$$\text{Now } \pi s_n = \int_0^{\pi} \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} f(t)\, dt$$

$$= \int_0^{k/n} \sin nt \cot \tfrac{1}{2}t f(t) dt + \int_{k/n}^{\pi} \sin nt \cot \tfrac{1}{2} t f(t)\, dt$$

$$+ \int_0^{\pi} \cos nt\, f(t) dt.$$

Denoting the three integrals respectively by α_n, β_n, γ_n, we note at the outset that

$$\operatorname*{Lim}_{n=\infty} \gamma_n = 0,$$

so that

$$\left(\sum_0^n |\gamma_m|^q \right)^{\frac{1}{q}} < \epsilon n^{\frac{1}{q}}$$

for $n >$ a number n_1 (ϵ) dependent on ϵ.

Also, if $\eta = \frac{k}{n}$, and $\mathrm{F}(t) = \int_0^t f(t)\, dt$,

we have, integrating by parts,

$$\alpha_n = \sin n\eta \cot \tfrac{1}{2}\eta \mathrm{F}(\eta) - \int_0^{\eta} \left(n \cos nt \cot \tfrac{1}{2}t \right.$$

$$\left. - \tfrac{1}{2} \sin nt \operatorname{cosec}^2 \frac{t}{2} \right) \mathrm{F}(t) dt$$

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$$=o(1)+\int_0^{\eta} o\left\{\left(\frac{n}{t}+\frac{nt}{t^2}\right)t\right\}dt=o(1)\longrightarrow 0,$$

when k is fixed. Hence, when k is fixed, we have

$$\left(\sum_0^n \left| a_m \right|^q\right)^{\frac{1}{q}} < \epsilon n^{\frac{1}{q}}$$

for n greater than a definite number $n_2(\epsilon, k)$ dependent on ϵ and k.

(*b*) It can be shown that

$$\left(\sum_1^n \left| \beta_m \right|^q\right)^{\frac{1}{q}} < \epsilon_k n^{\frac{1}{q}},$$

where ϵ_k is a function of k only which tends to zero when $k\longrightarrow\infty$.

For, if $c_n(\tau)$ denote the nth Fourier sine co-efficient of the odd function $\chi(t)$ which is equal to $f(t)$ in $(0, \tau)$ and to zero in (τ, π),

$$\beta_m=\int_{\eta}^{\pi} \sin mt \cot\frac{t}{2} f(t)dt=\int_{\eta}^{\pi} \cot\frac{t}{2}\left(\frac{d}{dt}\int_0^t \sin mu f(u)du\right)dt$$

$$=-\tfrac{1}{2}\pi \cot\frac{\eta}{2} c_m(\eta)+\tfrac{1}{4}\pi\int_{\eta}^{\pi} \operatorname{cosec}^2 \tfrac{1}{2}t.\ c_m(t)dt.$$

Now from the above it follows, by Minkowski's inequality, that

$$\left(\sum_1^n \left| \beta_m \right|^q\right)^{\frac{1}{q}} \leq \frac{\pi}{2} \cot\frac{\eta}{2}\left(\sum_1^n \left| c_m(\eta) \right|^q\right)^{\frac{1}{q}}$$

$$+\frac{\pi}{4}\int_{\eta}^{\pi} \operatorname{cosec}^2 \tfrac{1}{2}t \left(\sum_1^n \left| c_m(t) \right|^q \right)^{\frac{1}{q}} dt.$$

Again, assume $q>2$, $q'=\frac{q}{q-1}<p$, this being legitimate because the conclusion is stronger the larger q is. Then by Hausdorff's inequality,*

$$\left(\sum_1^n \left| c_m(t) \right|^q \right)^{\frac{1}{q}} \leqq \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \left| \chi(u) \right|^{q'} du \right)^{\frac{1}{q'}}$$

$$= \left(\frac{1}{\pi}\int_{-t}^{t} \left| f(u) \right|^{q'} \right) du \Big)^{\frac{1}{q'}} \leq A t^{\frac{1}{q'}},$$

where A is a constant, since the first condition of the theorem is satisfied a fortiori when p is replaced by the smaller index q'. Hence

$$\left(\sum_1^n \left| \beta_m \right|^q \right)^{\frac{1}{q}} \leqq \frac{A}{\eta}.\eta^{\frac{1}{q'}} + A\int_{\eta}^{\pi} \frac{1}{t^2}.t^{\frac{1}{q'}}\, dt \leq A\eta^{-\frac{1}{q}}$$

$$= A k^{-\frac{1}{q}} n^{\frac{1}{q}} = \epsilon_k n^{\frac{1}{q}}.$$

* $\left[\sum_{\lambda=0}^{\infty} |a_\lambda|^s \right]^{\frac{1}{s}} \leq \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^r \right]^{\frac{1}{r}} dt,$

where a_λ is the typical Fourier constant of a function $f(x)$ whose rth power is summable, $\frac{1}{r}+\frac{1}{s}=1$ and $s>r$. (See Hausdorff's paper, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen" in *Math. Zeit.*, Bd. 16, pp. 163-169).

(*c*) Using the results obtained in (*a*) and (*b*), we have

$$\pi\Big(\sum_1^n |s_m|^q\Big)^{\frac{1}{q}} \leq \Big(\sum_1^n |\alpha_m|^q\Big)^{\frac{1}{q}}$$

$$+\Big(\sum_1^n |\beta_m|^q\Big)^{\frac{1}{q}} + \Big(\sum_1^n |\gamma_m|^q\Big)^{\frac{1}{q}} < (\epsilon_k + 2\epsilon) n^{\frac{1}{q}}$$

if n is greater than n_1 or n_2, and our conclusion follows by choice first of k and then of n.

§ 30.

49. I will conclude this lecture by stating the criticism of G. Prasad of the various criteria; his criticism may be briefly summed up in the following statements :—

(*i*) Carleman's criterion can *never* be satisfied at a point x_0 where $f(x)$ has such a discontinuity of the second kind that

$$\phi(t) = \chi(t) \cos \psi\ (t),$$

χ and ψ being both monotone, and $\psi(t)$ being unbounded, in the neighbourhood of $t=0$; notwithstanding this, there is strong summability in a number of cases, *e.g.*, when χ is bounded and $\psi(t) \asymp \log \frac{1}{t^2}$, so that Carleman's criterion *fails* in these cases.

(*ii*) If χ is unbounded, the second criterion of Hardy and Littlewood, which is the most comprehensive of the hitherto known criteria, can never be satisfied; notwithstanding this, there is strong summability in a number of cases, *e.g.*, when $\chi = t^{-\frac{2}{5}}$, $\psi = t^{-1}$, so that the criterion of Hardy and Littlewood fails in this case. A similar remark holds true in the case of Sutton's generalization of Carleman's criterion.

(*iii*) Very probably, much less restrictive sufficient conditions for strong summability can be found similar to those given by G. Prasad for ordinary summability and stated in Art. 37.

The following considerations will make clear the truth of the above statements :—

(*a*) From Art. 36 it is clear that, for $\phi(t)=\cos\frac{1}{t}$,

$$\int_0^\delta |\phi(t)|\,dt=\frac{2}{\pi}\delta+0(\delta).$$

Thus Carleman's condition

$$\int_0^\delta |\phi(t)|\,dt=0(\delta)$$

is *not* satisfied. Similarly, as shown in G. Prasad's paper on the failure of Lebesgue's criterion for summability (C1), it can be proved that generally for $\psi(t) \asymp \log\frac{1}{t^2}$ and χ bounded,

$$\int_0^\delta |\phi(t)|\,dt \text{ is } \bigcirc(\delta) \text{ and } \textit{not } 0(\delta).$$

Notwithstanding this, Hardy and Littlewood's criterion is satisfied, and therefore it is proved that Carleman's criterion fails when χ is bounded and $\psi \asymp \log\frac{1}{t^2}$.

(*b*) It is known * that

$$\int_0^\delta |\phi(t)|\,dt \asymp \delta,$$

when χ is unbounded, $\psi \asymp \log\frac{1}{t^2}$ and $\frac{\chi}{\psi'} \prec t$.

* See my paper in the *Bulletin of the Calcutta Mathematical Society*, Vol. XIX, pp. 1-12.

Thus Hardy and Littlewood's condition

$$\int_0^{\delta} | \phi(t) |^p dt = O(\delta), (p > 1)$$

can never be satisfied in this case.

Notwithstanding this, it can be shown, by modifying the proof of Hardy and Littlewood, that, for $\chi = t^{-\frac{2}{5}}$, $\psi = t^{-1}$, $\sum_0^n | s_m - s |^3 = 0(n)$ and that, consequently, there is strong summability.

SIXTH LECTURE.

Properties of Fourier co-efficients ; question of uniqueness of Fourier expansion ; and other questions.

§ 31.

50. In to-day's lecture, I will deal chiefly with the properties of the Fourier constants or Fourier co-efficients corresponding to a given function $f(x)$; these, as already stated on several occasions, are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \cos n\alpha \, d\alpha,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \sin \alpha \, d\alpha,$$

$$n = 0, 1, 2, \ldots\ldots\ldots\ldots$$

I proceed to consider first the behaviour of these co-efficients with increasing n, assuming, unless the contrary is explicitly stated, that the notion of integration is that of Riemann.

(a) It follows from Theorem (T_1) of the second lecture that

$$\lim_{n=\infty} a_n = 0, \qquad \lim_{n=\infty} b_n = 0,$$

when $f(x)$ is bounded.

(b) It is easy to see that the above holds even if $f(x)$ is unbounded provided that $|f(x)|$ is integrable, the proof for the case in (a) being applicable with slight modifications to this case.

(c) As stated by Riemann, when $f(x)$ is integrable but not $|f(x)|$, a_n and b_n need not tend to zero with increasing n. On account of the historical importance of Riemann's proof of his statement,* I reproduce it here almost word by word.

* "Ueber die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe," §XIII.

Proof:

Let

$$\int f(x)\,dx=\phi(x)\cos\psi(x),$$

by assuming $\phi(x)$ to be infinitely small and $\psi(x)$ to be infinitely large as x tends to 0, these functions being always continuous together with their differential co-efficients and not having an infinite number of maxima and minima.

Then we have

$$f(x)=\phi'(x)\cos\psi(x)-\phi(x)\,\psi'(x)\sin\psi(x),$$

and

$$\int f(x)\cos n(x-a)\,dx$$

shall be equal to the sum of four integrals,

$$\frac{1}{2}\int \phi'(x)\cos\{\psi(x)\pm n(x-a)\}\,dx,$$

$$-\frac{1}{2}\int \phi(x)\psi'(x)\sin\{\psi(x)\pm n(x-a)\}\,dx.$$

$\psi(x)$ being supposed to be positive, let us consider the term

$$-\frac{1}{2}\int \phi(x)\,\psi'(x)\sin\{\psi(x)+n(x-a)\}\,dx,$$

and investigate in the interval of integration the place where the change of sign of the sine is slowest. If we put

$$\psi(x)+n(x-a)=y,$$

then this change will take place in the neighbourhood of the value of x where

$$\frac{dy}{dx}=0,$$

and, consequently,

$$\psi'(a)+n=0,$$

substituting a for x.

Let us study the behaviour of the integral

$$-\frac{1}{2}\int_{a-\epsilon}^{a+\epsilon}\phi(x)\,\psi'(x)\sin y\,dx$$

for the case when ϵ is infinitely small and n is infinitely large; also introduce y as the variable of integration.

Putting

$$\psi(a)+n(a-a)=\beta,$$

we have, for sufficiently small ϵ,

$$y=\beta+\psi''(a).\frac{(x-a)^2}{2!}+\ldots\ldots;$$

and $\psi''(a)$ is positive because $\psi(x)$ is infinitely large and positive for infinitely small x.

Therefore

$$\frac{dy}{dx}=\psi''(a)(x-a)=\pm\sqrt{2\psi''(a)(y-\beta)},$$

according as

$$x>a \text{ or } x<a;$$

and

$$-\frac{1}{2}\int_{\alpha-\epsilon}^{\alpha+\epsilon}\phi(x)\,\psi'(x)\sin y\,dy$$

$$=\frac{1}{2}\left\{\int_{\beta+\psi'(\alpha).\frac{\epsilon^2}{2}}^{\beta}-\int_{\beta}^{\beta+\psi''(\alpha).\frac{\epsilon^2}{2}}\right\}\frac{\sin y\,dy}{\sqrt{y-\beta}}.\frac{\phi(\alpha)\,\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}$$

$$=-\int_{0}^{\psi''(\alpha).\frac{\epsilon^2}{2}}\sin(y+\beta)\,\frac{dy}{\sqrt{y}}.\frac{\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}.$$

If the quantity ϵ is decreased with increasing n in such a manner that $\psi''(\alpha).\epsilon^2$ becomes infinitely large; then, by supposing that

$$\int_{0}^{\infty}\sin(y+\beta)\,\frac{dy}{\sqrt{y}},$$

which is, as known, equal to $\sqrt{\pi}\sin\left(\beta+\frac{\pi}{4}\right)$, is not zero, and leaving aside the negligible quantities, we have

$$-\frac{1}{2}\int_{\alpha-\epsilon}^{\alpha+\epsilon}\phi(x)\psi'(x)\sin\{\psi(x)+n(x-\alpha)\}dx$$

$$=-\sin\left((\beta+\frac{\pi}{4}\right)\frac{\sqrt{\pi}\,\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}.$$

Now it is easily seen that, if $\frac{\psi(x)}{log\ x}$ becomes infinitely large for x infinitely small, one can always choose $\phi(x)$ in such a manner that $x\phi(x)$ becomes infinitely small with x and

$$\frac{\phi(x)\psi'(x)}{\sqrt{2\psi''(x)}} = \frac{\phi(x)}{\sqrt{-2\frac{d}{dx}\frac{1}{\psi'(x)}}} = \frac{\phi(x)}{\sqrt{-2\ lim\ \frac{1}{x\psi'(x)}}}$$

becomes infinitely large.

Hence the integral $\int_{-\pi}^{\pi} f(x)dx$ has a meaning, although

$$\int_{-\pi}^{\pi} f(x)|\cos n(x-a)\,dx$$

may not be infinitely small with increasing n.

Example:

For $f(x)=\frac{d}{dx}\left(x^{\nu}\ \cos\frac{1}{x}\right)$, $0<\nu<\frac{1}{2}$,

$\int_{-\pi}^{\pi} f(x)\cos n(x-a)dx$ behaves as

$$\frac{1}{2}\sin\left(2\sqrt{n}\ -na+\frac{\pi}{4}\right)\sqrt{\pi}\ n^{\frac{1-2\nu}{4}}.$$

Thus in this case a_n and b_n are both unbounded for unbounded n.

51. (a) If $f(x)$ is bounded and monotone in the interval $(-\pi, \pi)$ then

$$\left|\ a_n\ \right| < \frac{K}{n}\ ,\ \left|\ b_n\ \right| < \frac{K}{n}\ ,$$

where K is a fixed number depending on the upper bound of $f(x)$ in $(-\pi, \pi)$.

Proof:

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = f(-\pi+0)\int_{-\pi}^{\alpha}\cos nx\,dx + f(\pi-0)\int_{\alpha}^{\pi}\cos nx\,dx$$

by the second mean-value theorem.

Hence $\left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right| < \frac{K}{n}$; similarly for $\left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right|$.

(*b*) If $f(x)$ is of bounded variation in the interval $(-\pi, \pi)$, then

$$a_n = O\left(\frac{1}{n}\right),\ b_n = O\left(\frac{1}{n}\right).$$

For, $f(x)=f_1(x)-f_2(x)$, where $f_1(x)$ as well as $f_2(x)$ is monotone.

(*c*) It is not necessary that for a function of bounded variation

$$\operatorname*{Lim}_{n=\infty} na_n = 0,\quad \operatorname*{Lim}_{n=\infty} nb_n = 0.$$

This is clear from the following examples:

(1) If $f(x) = +\left(\frac{\pi-x}{2}\right)$ or $-\frac{\pi+x}{2}$ according as x is positive or negative; then $a_n=0$, $b_n=\frac{1}{n}$.

(2) If $f(x)=-\frac{\pi}{4}$ or $+\frac{\pi}{4}$ according as x is negative or positive, then $a_n=0$, $b_{2k}=0$, $b_{2k+1}=\frac{1}{2k+1}$.

(3) If

$$f(x) = -\frac{\pi\sqrt{2}}{4} \text{ for } x < -\frac{3\pi}{4} \text{ and for } x > \frac{\pi}{4},$$

$$f(x) = \frac{\pi\sqrt{2}}{4} \text{ for } -\frac{3\pi}{4} < x < \frac{\pi}{4};$$

$$n\,a_n = 0, 1, 0, 1, 0, -1, 0, -1,$$

$$n\,b_n = 0, -1, 0, 1, 0, 1, 0, -1,$$

according as

$$n = 0, 1, 2, 3, 4, 5, 6, 7 \pmod 8.$$

(*d*) By giving the following example, F. Riesz * has proved that even for a *continuous* function of bounded variation $n\,a_n$ and $n\,b_n$ may not both tend to 0 with $\frac{1}{n}$.

Example:

$$\text{If } f(x) = -x + \lim_{m=\infty} \int_0^x (1+\cos x)\,(1+\cos 4x) \times$$

$$(1+\cos 4^2 x)\ldots(1+\cos 4^{m-1} x)\,dx,$$

then $n\,a_n = 0$ for every n but $n\,b_n = 1$ for every n of the form 4^k, k being any positive integer.

(*e*) Neder† has proved that if a function of period 2π and of bounded variation is such that

$$\lim_{n=\infty} n\,a_n = 0 \text{ and } \lim_{n=\infty} n\,b_n = 0$$

then it can have only removeable discontinuities.

52. I proceed now to prove that

$$\lim_{n=\infty} a_n = 0, \quad \lim_{n=\infty} b_n = 0,$$

when $f(x)$ is integrable in the sense of Lebesgue.

* "Ueber die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung" (*Mathematische Zeitschrift*, Bd, 2, 1918). As pointed out by Riesz, even a continuous function of bounded variation is not necessarily the indefinite integral of its differential co-efficient.

† *Math. Zeit.*, Bd. 6. p. 270.

Proof :

By definition

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt. \qquad \ldots \quad (1)$$

Hence

$$a_n = \frac{1}{\pi} \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f\left(u+\frac{\pi}{n}\right) \cos n\left(u+\frac{\pi}{n}\right) du$$

$$= -\frac{1}{\pi} \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f\left(u+\frac{\pi}{n}\right) \cos nu \, du$$

$$= -\frac{1}{\pi} \int_{-\pi}^{\pi} f\left(u+\frac{\pi}{n}\right) \cos nu \, du, \qquad \ldots \quad (2)$$

as the integrand is periodic with period 2π,

Adding the equations (1) and (2), we have

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ f(u) - f\left(u+\frac{\pi}{n}\right) \right\} \cos nu \, du$$

Similarly

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ f(u) - f\left(u+\frac{\pi}{n}\right) \right\} \sin nu \, du.$$

Thus

$$|a_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(u+\frac{\pi}{n}\right) - f(u) \right| du,$$

$$|b_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(u+\frac{\pi}{n}\right) - f(u) \right| du.$$

But it is a property* of Lebesgue's integral that

$$\lim_{\tau=0} \int_{-\pi}^{\pi} |f(u+\tau)-f(u)|\, du = 0.$$

* This property may be established as follows :

(*a*) For a prescribed $\epsilon > 0$ a bounded measureable function $g(u)$ can be found so that

$$\int_{-\pi}^{\pi} |f-g|\, du < \frac{\epsilon}{2}. \qquad \ldots \quad (i)$$

(*b*) g can be always approximated to by a sequence of functions $\chi_1, \chi_2, \ldots$ so that $\lim_{n=\infty} \chi_n = g$ and no $|\chi_n|$ is greater than the upper limit of $|g|$. Thus

$$\lim_{n=\infty} \int_{-\pi}^{\pi} |g-\chi_n|\, du = 0$$

and, consequently, a function χ_ν exists such that

$$\int_{-\pi}^{\pi} |g-\chi_\nu|\, du < \frac{\epsilon}{2}. \qquad \ldots \quad (ii)$$

Therefore $\int_{-\pi}^{\pi} |f-\chi_\nu|\, du < \epsilon$ from (*i*) and (*ii*); and corresponding to a prescribed quantity $\delta > 0$ it is always possible to find a function χ such that

$$\int_{-\pi}^{\pi} |f-\chi|\, du < \frac{\delta}{3}. \qquad \ldots \quad (iii)$$

Hence, putting $\tau = \frac{\pi}{n}$, we have

$$\lim_{n=\infty} | a_n | = 0, \quad \lim_{n=\infty} b_n = 0.$$

§ 32.

53. Probably the most important properties of Fourier co-efficients are contained in the following theorems :

If $f(x)$ be a function such that its square is integrable, and a_0, a_1, b_1 ...are the Fourier co-efficients corresponding to it, then the series $\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ converges to $\frac{1}{\pi} \int_{-\pi}^{\pi} \{f(x)\}^2 dx$. ...($T_9$)

If the squares of the co-efficients of a trigonometric series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin x)$$

form a convergent series

$$\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

then the a's and b's are the Fourier co-efficients corresponding to a function whose square is integrable. ... (T_{10})

(*c*) Now for sufficiently small values of t the equation $\chi(u+t) = \chi(u)$ is always fulfilled, excepting certain finite number of points in whose neighbourhood $\chi(u)$ has a saltus. Thus t_0 can be chosen so that for $t \leq t_0$,

$$\int_{-\pi}^{\pi} | \chi(u+t) - \chi(u) | \, du < \frac{\delta}{3} . \quad \ldots \quad (iv)$$

(*d*) Hence $\int_{-\pi}^{\pi} | f(u+t) - f(u) | \, du < \delta$ by using (*iii*) and (*iv*). Therefore

$$\lim_{t=0} \int_{-\pi}^{\pi} | f(u+t) - f(u) | \, du = 0,$$

The first theorem is named after Parseval; * and the second, which is the converse of the first, is called Riesz-Fischer theorem, as it was given independently and almost at the same time by F. Riesz and E. Fischer.†

54. *Proof of* (T_9):

(*a*) As proved in Art. 33 of the fourth lecture,

$$S_n(x)-f(x)=\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}\phi(z)\left(\frac{\sin nz}{\sin z}\right)^2 dz,$$

S_n being the Cesàro mean and ϕ standing for

$$f(x+2z)+f(x-2z)-2f(x).$$

Now apply Schwarz's inequality‡ to the above integral. Then

$$(S_n-f)^2 \leq \frac{1}{n\pi}\int_0^{\frac{\pi}{2}}\phi^2\left(\frac{\sin nz}{\sin z}\right)^2 dz.\ \frac{1}{n\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin nz}{\sin z}\right)^2 dz.$$

* This theorem is due to Parseval (*Paris mém. prés.* I, 1806, p. 638) who, however, did not give any rigorous proof. In fairly recent years, proofs have been given by Hurwitz (*Math. Ann.* Bd. 57) and others. In the proof given here, the notion of integration is Lebesgue's.

† Fischer communicated his theorem to the Mathematical Society of Brünn on the 5th March, 1907, and Riesz's paper was communicated by Hilbert to the Society of Sciences of Göttingen on the 13th March, 1907. But Fischer published his result after Riesz had published his result.

‡ Denoting $\int_a^b F(x).\,G(x)\,dx$ by (F, G); Schwarz's inequality is

$$(F, G)^2 \leq (F, F).\,(G, G),$$

In the present case, $F=\phi\,\frac{\sin nz}{\sin z}$, $G=\frac{\sin nz}{\sin z}$.

But it is known that

$$\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin nz}{\sin z}\right)^2 dz=\frac{1}{2}.$$

Therefore

$$(S_n-f)^2 \leq \frac{1}{2n\pi}\int_0^{\frac{\pi}{2}}\phi^2.\left(\frac{\sin nz}{\sin z}\right)^2 dz;$$

hence

$$\int_{-\pi}^{\pi}(S_n-f)^2dx \leq \frac{1}{2\pi n}\int_0^{\frac{\pi}{2}}\psi(z)\left(\frac{\sin nz}{\sin z}\right)^2 dz, \quad (1)$$

where $\psi(z)=\int_{-\pi}^{\pi}\phi^2 dx.$

Now from the fact that f^2 is integrable, it can be deduced without difficulty, as shown in the foot-note of Art. 53, that

$$\lim_{z=0}\int_{-\pi}^{\pi}\{f(x+2z)-f(x)\}^2dx=0, \quad \ldots \quad (2)$$

$$\lim_{z=0}\int_{-\pi}^{\pi}\{f(x-2z)-f(x)\}^2dx=0.$$

But $\phi^2=[\{f(x+2z)-f(x)\}+\{f(x-2z)-f(x)\}]^2 \leq \{f(x+2z)-f(x)\}^2 + \{f(x-2z)-f(x)\}^2.$

Hence, applying (2), we have $\lim_{z=0}\psi(z)=0$, and, consequently,

$$\lim_{z=0}\frac{1}{z}\int_0^z\psi(z)dz=0.$$

Therefore, from Art. 34,

$$\operatorname*{Lim}_{n=\infty} \frac{1}{2n\pi} \int_0^{\frac{\pi}{2}} \psi(z) \left(\frac{\sin nz}{\sin z} \right)^2 dz = 0 ;$$

and, consequently from (1) it follows that

$$\operatorname*{Lim}_{n=\infty} \int_{-\pi}^{\pi} (S_n - f)^2 \, dx = 0. \qquad \ldots \quad (3)$$

(*b*) But if t_n denote any trigonometrical polynomial of the *nth* degree, it follows from the definition of the Fourier constants that

$$\int_{-\pi}^{\pi} (f - s_n) \, t_n \mathrm{d}x = 0 ;$$

and, as special cases of the above,

$$\int_{-\pi}^{\pi} (f - s_n) \, s_n \, dx = 0.$$

$$\int_{-\pi}^{\pi} (f - s_n) \, S_n \, dx = 0.$$

Now, from the identity,

$$(f - S_n)^2 = (f - s_n)^2 + (s_n - S_n)^2 - 2 \, (f - s_n) \, S_n + 2 \, (f - s_n) \, s_n,$$

it follows that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (f - S_n)^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (f - s_n)^2 \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} (s_n - S_n)^2 dx.$$

Therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f-S_n)^2dx \geq \frac{1}{\pi}\int_{-\pi}^{\pi}(f-s_n)^2dx \equiv r_n \text{ say.} \quad \ldots \quad (4)$$

Again it is clear from the definitions of s_n and the Fourier coefficients, that

$$r_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2dx-\left\{\tfrac{1}{2}a_0^2+\sum_{k=1}^{n}(a_k^2+b_k^2)\right\},$$

whatever finite quantity n may be.

But

$\lim_{n=\infty} r_n$ is 0, because of (3) and (4).

Therefore it is proved that

$$\tfrac{1}{2}a_0^2+\sum_{1}^{\infty}(a_n^2+b_n^2) \text{ is convergent to } \frac{1}{\pi}\int_{-\pi}^{\pi}f^2dx.$$

§ 33.

55. *First Proof of* T_{10}:

(*a*) Consider the trigonometric series

$$\sum_{1}^{\infty}\frac{a_n \sin nx-b_n \cos nx}{n} \quad \ldots \quad (1)$$

Then

$$\left|\frac{a_k \sin kx-b_k \cos kx}{k}\right| \leq \frac{(a_k^2+b_k^2)^{\frac{1}{2}}}{k};$$

also it follows from the well-known inequality,

$$\left(\sum_{k=1}^{n}A_k B_k\right)^2 \leq \sum_{k=1}^{n}A_k^2 \cdot \sum_{k=1}^{n}B_k^2,$$

the A's and B's being any real quantities,

that

$$\sum_{k=1}^{n} \frac{(a_k^2 + b_k^2)^{\frac{1}{2}}}{k} \leq \left\{ \sum_{k=1}^{n} (a_k^2 + b_k^2) \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{k=1}^{n} \frac{1}{k^2} \right\}^{\frac{1}{2}}.$$

But $\sum_{1}^{\infty} \frac{1}{n^2}$ is convergent and, by hypothesis, $\sum_{1}^{\infty} (a_n^2 + b_n^2)$ is convergent.

Therefore the series (1) is convergent for every value of x, and its partial sum

$$T_n = \sum_{k=1}^{n} \frac{a_k \sin kx - b_k \cos kx}{k} \text{ is numerically less than}$$

$$\left\{ \sum_{1}^{\infty} (a_k^2 + b_k^2) \cdot \sum_{1}^{\infty} \frac{1}{k^2} \right\}^{\frac{1}{2}}$$

whatever value n may have; let $\operatorname{Lim}_{n=\infty} T_n$ be denoted by $F(x)$.

(*b*) Now, for any sub-interval $I=(\alpha, \beta)$ of the interval $I_0=(-\pi, \pi)$, introduce the interval-functions $\Phi(I)$, $\Phi_n(I)$, where

$\Phi(I)=F(\beta)-F(\alpha)$, $\Phi_n(I)=T_n(\beta)-T_n(\alpha)$, so that

$$\operatorname{Lim}_{n=\infty} \Phi_n = \Phi.$$

But, obviously,

$$\Phi_n(I) = \int_{\alpha}^{\beta} D\Phi_n dx = \int_{\alpha}^{\beta} s_n dx = \int_{I} s_n dx.$$

Hence, because of Schwarz's inequality,

$$\{\Phi_n(I)\}^2 \leq \int_{I} s_n^2 dx \cdot mI, \qquad \ldots (2)$$

mI denoting the measure of I.

As (2) is true for all the non-overlapping sub-intervals I_1, I_2,...I_p of I_0,

$$\sum_{k=1}^{p} \frac{\{\Phi_n(I_k)\}^2}{mI_k} \leq \sum_{k=1}^{p} \int_{I_k} s_n^2 \, dx \leq \int_{-\pi}^{\pi} s_n^2 \, dx = \pi \sum_{k=1}^{n} (a_k^2 + b_k^2).$$

But by hypothesis

$$\pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

is convergent; let the sum be denoted by G.

Therefore

$$\sum_{k=1}^{p} \frac{\{\Phi_n(I_k)\}^2}{mI_k} \leq G;$$

and, consequently,

$$\sum_{k=1}^{p} \frac{\{\phi(I_k)\}^2}{mI_k} \leq G.$$

(c) From the above result it can be deduced without much difficulty that

$$\Phi = \int f(x) \, dx,$$

where f is a function which is integrable with its square.* Hence the function $F(x)$ is absolutely continuous and almost everywhere

$$DF = f.$$

Also the series (1) may be integrated term by term. Therefore

$$-\frac{b_k}{k} = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos kx \, dx, \quad \frac{a_k}{k} = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin kx \, dx.$$

Since $F(-\pi) = F(\pi)$, applying integration by parts, we have

* See the paper of Riesz in *Math. Ann.*, Bd. 69, 1910.

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx,\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx);$$

thus it is proved that the a's, and b's, are Fourier co-efficients corresponding to a function whose square is integrable.

56. *Second Proof of* T_{10} :

This proof is based on the following result * due to Riesz :

If the succession of functions $f_n(x)$, whose pth powers are summable $(0 < p)$, is such that, for a given interval (α, β),

$$\operatorname*{Lim}_{m\to\infty,\, n\to\infty}\int |f_m-f_n|^p\,dx=0\ (0<p),$$

then the succession defines a function $f(x)$ such that $(a)\ |f_n-f|<\epsilon$, an arbitrarily small but positive quantity, except at a set whose measure vanishes when $n\to\infty$, and (b) for every sub-interval (α_1, β_1) of (α, β)

$$\operatorname*{Lim}_{n\to\infty}\int_{\alpha_1}^{\beta_1} |f-f_n|^q\,dx=0\ (0<q\leq p).$$

Now we know that, as $\int_{-\pi}^{\pi}(s_n-s_m)^2dx=\pi\sum_{k=n+1}^{m}(a^2{}_k+b^2{}_k)$,

assuming that $m > n$, and as by hypothesis $\sum_{k=1}^{\infty}(a_k{}^2+b_k{}^2)$

is convergent,

$$\operatorname*{Lim}_{m\to\infty,\, n\to\infty}\int_{-\pi}^{\pi}(s_n-s_m)^2dx=0.$$

* *Comptes Rendus*, Vol. 148, 1909.

Hence it follows from the above theorem that there is a function $f(x)$, whose square is summable, such that

$$\lim_{n\to\infty} \int | f-s_n | \, dx=0,$$

and, consequently,

$$\lim_{n\to\infty} \int (f-s_n)dx=0.$$

Hence the series

$$\sum_1^\infty \frac{a_k \sin kx-b_k \cos kx}{k} = \int f(x)dx=\mathrm{F}(x).$$

Therefore, proceeding as in (*c*) of Art. 55, we have a_k, b_k as the Fourier co-efficients of $f(x)$.

§ 34.

57. The following theorems (T_{11}) and (T_{12}) are probably the widest generalizations * respectively of (T_9) and (T_{10}); they have been recently further generalized by Pollard † by the use of Denjoy integration.

If $p\geq 2$ and $\frac{1}{p}+\frac{1}{q}=1$, and, further, $\int_{-\pi}^{\pi} | f(x) |^q dx<\infty$; then

$$\left(\frac{a_0}{\sqrt{2}} \right)^p + \sum_{n=1}^\infty (| a_n |^p + | b_n |^p)<\infty,$$

and $\left\{ \left| \frac{a_0}{\sqrt{2}} \right|^p + \sum_1^\infty (| a_n |^p + b_n |^p) \right\}^{\frac{1}{p}}$

$$\leq \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} | f(x) |^q dx \right\}^{\frac{1}{q}} . \quad (\mathrm{T}_{11})$$

* F. Riesz, *Math. Zeit.*, Vol. 18, 1923.

† ' On the generalizations of the theorems of Parseval and Riesz-Fischer" (*Proc. Cambridge Phil. Soc.*, 1927).

If $p \leq 2$ and $\frac{1}{p}+\frac{1}{q}=1$, and, further,

$$\left| \frac{a_0}{\sqrt{2}} \right|^p + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p) < \infty,$$

then there is a function $g(x)$ such that

$$\int_{-\pi}^{\pi} |g(x)|^q < \infty,$$

and, for $n=0, 1, 2, \ldots$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx \, dx, \; b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx \, dx.$$

Moreover,

$$\left\{ \left| \frac{a_0}{\sqrt{2}} \right|^p + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p) \right\}^{\frac{1}{p}}$$

$$\geq \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} |g(x)|^q \, dx \right\}^{\frac{1}{q}} . \qquad (T_{13}).$$

58. Another direction in which the theorem (T_9) admits of generalization gives the following theorems * (T_{13}), (T_{14}) and (T_{15}):

If $\{f(x)\}^2$, $\{g(x)\}^2$ are each integrable, then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, g(x) dx = \frac{1}{2} a_0 \, a_0 + \sum_{n=1}^{\infty} (a_n a_n + b_n \beta_n),$$

the a's and b's being the Fourier co-efficients of $f(x)$ and the α's and β's those of $g(x)$. (T_{13}).

* See Hobson's "Theory of Functions of a Real Variable,..." pp. 575, 597 and 608.

If $\{f(x)\}^p$, $\{g(x)\}^q$ are both integrable in the interval $(-\pi, \pi)$, where p and q are positive numbers such that $\frac{1}{p}+\frac{1}{q}=1$, then the series

$$\frac{1}{2}\,a_0\alpha_0+\sum_{n=1}^{\infty}(a_n\alpha_n+b_n\beta_n)$$

converges (C1) to the value $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)dx$, and if the series be convergent, its sum has this value. (T_{14}).

If p, q be positive numbers such that $pq<1$, $\frac{(1+p)(1+q)}{1-pq}\leq 2$, the series

$$\left(\frac{a_0\alpha_0}{\sqrt{2}}\right)^{k+1}+\sum_{1}^{\infty}\{\,|\,a_n\alpha_n+b_n\beta_n\,|^{k+1}+|\,a_n\beta_n-b_n\alpha_n\,|^{k+1}\}$$

converges to a sum

$$\leq \frac{1}{\pi^k}\left\{\int_{-\pi}^{\pi}dx\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\,g(t)dt\right|^{1+\frac{1}{k}}\right\}^k,$$

where $k=\frac{1-pq}{p+q+2pq}$, provided that $|f(x)|^{1+p}$, $|g(x)|^{1+q}$ are integrable. (T_{15}).

59. The following examples * are intended to illustrate the bounds of our knowledge of the properties of Fourier co-efficients :

(1) If $f(x)$ is a bounded function which is continuous, or whose discontinuities, if any, are of the first kind, then $\sum_{2}^{\infty}\frac{a_n}{\log n}$ converges.

* (1), (3), (4) are due to W. H. Young (*Proc. L.M.S.*, Ser. 2, Vol. 12, pp. 41-70); (2) is due to Fatou (*Acta Math.*, Vol. 30) and Hardy (*Am. Math. Soc. Trans.*, Vol. 17); (5) is due to W. H. Young (*Proc. L.M.S.*, Ser. 2, Vol. 11, p. 423), and for (6) see Hobson's "Theory," p. 600.

(2) If $|f(x+\delta)-f(x)| < A|\delta|^k$, $(0 < k \leq 1)$: then, for $k=1$,

$$\operatorname*{Lim}_{n=\infty} na_n = \operatorname*{Lim}_{n=\infty} nb_n = 0, \text{ and } \sum_1^\infty n^2(a_n^2+b_n^2) \text{ is convergent.}$$

For $k<1$, $\sum_{n=1}^{\infty} n^{2(k-\epsilon)}(a_n^2+b_n^2)$ is convergent, if ϵ is arbitrarily small but positive.

(3) $\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}$ is a Fourier series and has for its sum

$$-\frac{1}{2}\cos x-(\sin x)(\pi-x)+\frac{\pi}{2}\int_0^\infty \mathrm{P}\, dy, \text{ P being}$$

$$\cosh y\{(\pi-x)\}/\sinh \pi y\left[\left\{\frac{\pi^2}{4}+(\log y)^2\right\}\right], \quad (0\leq x\leq \pi).$$

(4) $\sum_2^\infty \frac{\sin nx}{\log n}$, although convergent, is not a Fourier series, its sum being not integrable; as, in the neighbourhood of $x=0$, it behaves as $\frac{1}{x \log x}$.

(5) The necessary and sufficient condition that the series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

should be a Fourier series of some function $f(x)$, summable in $(-\pi, \pi)$, is that the integrated series should converge throughout the interval to a function which is the indefinite integral of a summable function.

(6) If $\sum_{n=1}^{\infty}\left(|a_n|^{1+\frac{1}{p}}+|b_n|^{1+\frac{1}{p}}\right)$ converges for a value of $p<1$, then the co-efficients are not necessarily Fourier co-efficients. For, as shown by Hardy and Littlewood, $\Sigma n^{-\alpha}\cos(n^2\pi x)$, $\Sigma n^{-\alpha}\sin(n^2\pi x)$, are not Fourier series for $\alpha \leq \frac{1}{2}$; and in fact no sum can be associated

with any of them. Similarly, Titchmarsh defines an odd function $f(x)$ to be $\log p . \cos(xp^2 \log p)$ for $\pi/\log(p+1) < x \leq \pi/\log p$, $(p=3, 4, 5, \ldots)$ and to be zero elsewhere, and shows that the series $\sum b_n \sin nx$ is not a Fourier series as the sum is not summable.

(7) Lebesgue* has shown that if $u(n)$ be a function which converges monotonously to 0 with $\frac{1}{n}$, then a continuous function $f(x)$, such that $|f(x)| \leq 1$, can be constructed for which the Fourier co-efficients are of order superior to that of $u(n)$.

§ 35.

60. (a) As regards the question, whether for a given function $f(x)$, there is only one series

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \quad \ldots (1)$$

corresponding to it; *i.e.*, whether the a's and b's are uniquely determined by $f(x)$, it is easily seen that the answer is in the affirmative.

(b) The next question, whether the Fourier series (1) defines uniquely the function $f(x)$ and does not correspond to a *different* function $\phi(x)$, is also to be answered in the affirmative :

Proof :

Let us consider $g(x) \equiv \int_{-\pi}^{x} f(x)dx - \frac{1}{2} a_0 x$. Then $g(x)$ is a function, which is periodic with period 2π, and continuous and of bounded variation. Therefore $g(x)$ is expansible in a Fourier series which converges uniformly to $g(x)$. Let this series be

$$\frac{1}{2} A_0 + \sum_1^\infty (A_n \cos nx + B_n \sin nx).$$

* *Annales sc. de l'école normale sup.*, 1903. p. 471

Then $A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx\, dx = -\frac{1}{n\pi} \int_{-\pi}^{\pi} \sin nx.\ Dg(x)dx,$

(by integration by parts)

Dg being any one of the four derivates of g. Now, excepting the points of a set of measure 0, Dg equals each of the other three derivates and is therefere equal to $f(x) - \frac{1}{2} a_0$. Therefore $A_n = -\frac{b_n}{n}$. Similarly, $B_n = \frac{1}{n} a_n$. Thus it is proved that

$\frac{1}{2}A_0 + \sum_1^\infty \frac{a_n \sin nx - b_n \cos nx}{n}$ converges uniformly in any interval to the function

$$\int_{-\pi}^{x} f(x)dx - \tfrac{1}{2} a_0 x,\ A_0 \text{ standing for } \frac{1}{\pi} \int_{-\pi}^{\pi} g(x)dx.$$

Therefore, for any interval (x_1, x_2),

$$\int_{x_1}^{x_2} f(x)dx = \frac{1}{2} a_0 (x_2 - x_1) + \sum_1^\infty \frac{a_n \sin nx_2 - b_n \cos nx_2}{n}$$

$$- \sum_1^\infty \frac{a_n \sin nx_1 - b_n \cos nx_1}{n}.$$

Similarly for $\int_{x_1}^{x_2} \phi(x)dx.$

Therefore, whatever x_1, x_2 may be, $\int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} \phi(x).$

Hence it follows that $f(x)$ equals $\phi(x)$, except at the points of a set of measure zero.

61. On account of its historical importance, the following question may be considered here. *Without postulating* that the series

$$\frac{1}{2}\ a_0 + \sum_1^\infty\ (a_n \cos nx + b_n \sin nx)$$

is a Fourier series, what can be said about its representing, in case it converges, its sum function uniquely ? In other words, are there any conditions under which a trigonometric series represents 0 without all of its co-efficients being 0 ?

(*a*) The first to attempt to answer these questions was G. Cantor* who proved first that the co-efficients must be all 0 if the series converges *everywhere* to 0, next that the co-efficients must be all 0 if the points where the series is not known to converge to 0 are finite in number, and lastly that the co-efficients must be all 0 even if such points form an infinite set provided the derived set consists of a finite number of points.

(*b*) F. Bernstein† has proved that the proviso that the set does not contain a perfect component suffices to ensure the vanishing of the co-efficients.

(*c*) D. Menchof‡ has given an example to show that the co-efficients of a trigonometric series *need* not be 0 if the set of points, at which the series is not known to converge, is a certain perfect set of measure 0.

(*d*) About the most recent publication on the subject is A. Zygmund's "Contribution á l' unicité du developpement trigonométrique" (*Math. Zeit.*, Vol. 24, 1926, pp. 40-46). According to Zygmund, Rajchman and Nina Bary obtained independently unenumerable sets of measure zero such that the co-efficients of the trigonometric series all vanish if the series converges to zero everywhere excepting at such a set. Such unenumerable sets contain perfect sets.

(*e*) If the series, for which a_n and b_n converge to zero, be such that its upper and lower sum-functions are summable in $(-\pi, \pi)$ and such that both of them are finite at every point, with the possible exception of a set which contains no perfect component, then the given series is the Fourier series corresponding to either the upper or lower sum-function of the series.

* *Crelle's Journal*, Vols. 72, 73, and *Math. Ann.*, Vol. 5.

† *Leipziger Berichte*, 1908.

‡ Comptes Rendus, 1916.

§ 36 ·

62. I will conclude this lecture by considering briefly the points of resemblance, and the points of difference, between the behaviour of the Fourier series of a function $f(x)$ and the behaviour of the Legendre or Bessel series corresponding to $f(x)$. For the sake of simplicity and fixity of ideas, it will be sufficient to confine our attention to Legendre's polynomials, the best known orthogonal functions after the sine and cosine functions.

The points in question are brought out by the following statements* in which the Legendre series is $\sum_0^\infty c_n \mathrm{P}_n(x)$, where

$$c_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\mathrm{P}_n(x)dx$$

or

$$\left(n+\frac{1}{2}\right) \int_0^\pi f(\cos\theta)\mathrm{P}_n(\cos\theta) \sin\theta \; d\theta.$$

(a) So far as ordinary convergence at a given point θ_0 in $(0, \pi)$ is concerned, the Legendre series converges, diverges or oscillates in the same manner as the Fourier cosine series of $f(\cos\theta)$ for the interval $(0, \pi)$.

* The most recent publications bearing on these statements are Fejér's papers "Ueber die Summabilität der Laplacesche Reihe durch arithmetische Mittel" and "Abschätzungen für die Legendreschen und verwandte Polynome" in *Math. Zeit.*, Bd. 24, 1926; F. Lukacs's paper "Ueber die Laplacesche Reihe" and Kogbetliantz's paper "Ueber die Summierbarkeit der Laplacesche Reihe" in *Math. Zeit.*, Bd. 14, 1922. Other papers on the subject are the classical paper of Fejér in *Math. Ann.*, Bd. 67, and his paper in *Rendiconti Cir. Mat. Palermo*, t. 38, and the papers of Haar (*Math. Ann.*, Bd. 69; *Rendiconti Cir. Mat. Palermo*, t. 32), Chapman (*Q. J. M.*, 1911; *Math. Ann.*, Bd. 72), Gronwall (*Math. Ann.*, Bde. 74, 75); Hilb (*Math. Zeit.* Bde. 5, 8), Volk (*Münchener Sitz.*, 1921), Plancherel (*Rendiconti Cir. Mat. di Palermo*, t. 33). In this connection may be also mentioned the papers of Kaczmarz (*Math. Zeit.*, Bd. 23), S.Borgen (*Math. Ann.*, Bd. 98), Hobson (*Proc. L.M.S.*, Ser. 2, Vols. 6, 7, 12), W. H. Young (*Proc. L.M.S.*, Ser. 2, Vol. 18), Mercer (*Phil. Trans.*, Vol. 211A), Weyl (*Math. Ann.*, Bd- 67), Rademacher (*Math. Ann.*, Bd. 68), Menchoff (*Fund. Math.*, Vol. 4). and Kogbetliantz (*Liouville's Journal*, 1924).

(*b*) In particular, all the known sufficient conditions of convergence of the Fourier cosine series relating to the nature of the function $f(\cos\theta)$ in an arbitrarily small neighbourhood of θ_0 are applicable without change as sufficient conditions for the convergence of the Legendre series.

(*c*) Also a *continuous* function can be found such that the corresponding Legendre series diverges at a point.

Proof of* (*c*)—

Let

$$\Phi(\theta) \text{ denote the function } \sum_{n=1}^{\infty} \frac{f_{\nu_n}(\theta)}{n^2}, \text{ where}$$

$$f_r(\theta)=\sqrt{r+1}\ \{P_r(\cos\theta)-P_{r+2}(\cos\theta)\}\ (1-\cos\theta) \text{ and } \nu_n=n^6.$$

Then, remembering that

$$\left|P_r(x)-P_{r+2}(x)\right| < \frac{A}{\sqrt{r+1}}, \ (-1\leq x\leq 1),$$

A being a finite quantity independent of r and x, it is obvious that

$$\left|\frac{f_{\nu_n}(\theta)}{n^2}\right| \leq \frac{2A}{n^2}.$$

Therefore $\Phi(\theta)$ is a continuous function, as the series defining it is absolutely and, consequently, uniformly convergent.

Now, by Christoffel's formula, the kth partial sum s_k of the Legendre series of $\Phi(\theta)$ at $\theta=0$ is

$$-\int_0^{\pi} \Phi(t).\ \frac{k+1}{2}\ \frac{P_{k+1}(\cos t)-P_k(\cos t)}{1-\cos t}\ \sin t\ dt$$

$$=\frac{k+1}{2}\sum_{n=1}^{\infty}\left[\frac{\sqrt{n^6+1}}{n^2}\int_{-1}^{1}\left\{P_{n^6}(x)-P_{n^6+2}(x)\right\}\{P_k(x)-P_{k+1}\}dx.\right.$$

* The example and proof are due to Lukacs (*Math. Zeit.*, Bd. 14).

Therefore

$$s_{n^6} = \frac{(n^6+1)^{\frac{3}{2}}}{n^2(2n^6+1)} > \frac{n}{2};$$

hence

$$\lim_{n=\infty} s_{n^6} = +\infty,$$

and, consequently, the Legendre series diverges at $\theta=0$.

(*d*)* So far as Cesàro summability at θ_0 is concerned, the analogue of the Cesàro mean of the first order of the Fourier series is not the Cesàro mean of the first order, but that of the *second* order, of the Legendre series.

(*e*)† In the case in which $f(x)$ is continuous at θ_0, the two series are summable (C1) at θ_0 to the function.

(*f*) In the case in which $f(x)$ has a discontinuity of the *second* kind at θ_0, a *general* treatment of the Legendre series similar to what is known of the Fourier series *is still a desideratum.*

* See Fejér's paper in *Math. Ann.*, Bd. 67, 1908.

† First proved by Haar (*Rend. Cir. Mat. Palermo*, t. 32, 1911).

APPENDIX A.

The Riemannian theory of trigonometric series.

1. In the Riemannian theory the central problem is, *not*, as in the ordinary theory of Fourier series, to find necessary or sufficient conditions for a given function $f(x)$ in order that the series corresponding to it be convergent, but to investigate the conditions which must be satisfied by the function representable by a *given trigonometric series*,

$$\tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx), \qquad \ldots \quad (A)$$

when the α's and β's satisfy the condition that, for increasing n, α_n and β_n are of a certain order of magnitude.

Riemann * considered the case for

$$\lim_{n=\infty} \alpha_n = 0, \quad \lim_{n=\infty} \beta_n = 0, \qquad \ldots \quad (B)$$

and established the following theorems :—

(I) Denoting by $F(x)$ the sum-function of the series,

$$\frac{\alpha_0}{4} x^2 - \sum_{n=1}^{\infty} \frac{1}{n^2} (\alpha_n \cos nx + \beta_n \sin nx), \qquad \ldots \quad (C)$$

$$\lim_{h=0} \frac{F(x+2h) - 2F(x) + F(x-2h)}{2h} = 0.$$

(II) If the given series (A) converges for a given value x_0 of x, then

$$\lim_{h=0} \frac{F(x_0+2h) - 2F(x_0) + F(x_0-2h)}{4h^2},$$

* "Ueber die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe," §VII and §VIII.

called $D^2F(x_0)$ or the generalized second differential co-efficient, exists and equals the sum of (A) at x_0.

Generally, $D^2F(x_0)$ is said to be the R-sum of the series at x_0. From these theorems, Riemann concludes that for a given function $f(x)$, representable by means of a series (A), for which (B) holds, in the sense that $f(x)$ is the second generalized differential co-efficient of (C), it is necessary and sufficient that

(1) $f(x)$ should be the second generalized differential co-efficient of a continuous function $F(x)$,

and (2) $$\lim_{\mu=\infty} \mu^2 \int_b^c F(t)\,\lambda(t)\cos\mu\,(t-x)dt=0,$$

uniformly with respect to x, for every arbitrary interval (b, c) and for every function $\lambda(t)$, which together with $\lambda'(t)$ vanishes at b and c and for which $\lambda''(t)$ is in (b, c) of bounded variation. (III).

It is of interest to note that if a series is summable (R) at a point, it is also summable (C, 3) at the same point.*

2. Another result of Riemann's may be given in the following form:—

Let ϵ be an arbitrarily chosen positive number less than $\frac{\pi}{2}$, and let $\rho(t)$ be a function defined in the interval $(-2\epsilon, 2\epsilon)$ of t, which has a bounded third differential coefficent. Let $\rho(t)$ have the value 1 in the whole interval $(-\epsilon, \epsilon)$ and the value zero at the points -2ϵ, 2ϵ. Then the necessary and sufficient condition that the series (A) with the condition (B) may be ordinarily convergent at the point x_0 is that

$$\frac{1}{2\pi}\int_{-2\epsilon}^{2\epsilon} F(t+x)\,\rho(t).\ \frac{d^2}{dt^2}\,\frac{\sin\,(n+\frac{1}{2})t}{\sin\,\frac{1}{2}t}\,dt \qquad \text{(IV)}$$

* This result was given by Rajchman (See *Fundamenta Math.*, 1922, p. 287).

should converge to a definite limit as $n \longrightarrow \infty$. From this it follows that the ordinary convergence of the series (A) at a point depends on the nature of the Riemann sum in a small neighbourhood of the point.

3. The following theorems are generalizations of Riemann's results and are due to A. Zygmund *:—

(V). Let the coefficients α_n, β_n of the series

$$\frac{1}{2}\alpha_0 + \sum_1^\infty (\alpha_n \cos nx + \beta_m \sin nx)$$

be $0\left(n^\gamma\right)$ where $\gamma > -1$ and let k be a positive integer such that $k-\gamma > 1$. Then, denoting by $F(x)$ the sum of the series integrated term by term k times, the uniform summability (C, γ) of the series in (a, b) depends only on the manner in which $F(x)$ behaves in $(a-\epsilon,\ b+\epsilon)$ whatever quantity $\epsilon > 0$ may be.

(VI) Let $\rho(t)$ be a function of period 2π, continuous with its successive derivatives of sufficiently high order. Further, let, for fixity of ideas, $0 \leqq a < b < 2\pi$, and $\rho(t)$ equal 0 outside (a, b) and be equal to 1 for $\alpha \leq v \leq \beta$, $(a < \alpha < \beta < b)$. Then the series

$$\frac{a_0}{2} + \sum_{m=1}^{n} (\alpha_m \cos mx + \beta_m \sin mx)$$

$$-\frac{(-1)^k}{2\pi}\int_a^b F(t)\rho(t)\,\frac{d^k}{dt^k}\left\{\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\right\}dt$$

$$(n=0,\ 1,\ 2,\ 3,\ldots)$$

is uniformly summable (C, γ) for $\alpha \leq x \leq \beta$, the generalized limit being zero.

* "Sur la théorie riemannienne des séries trigonométriques" (*Math. Zeit.*, Bd. 24, 1926).

4. On account of its historical interest, I proceed to reproduce the proof of (II) as given by Riemann :—

(*a*) It is easily seen that

$$\frac{F(x_0+2h)-2F(x_0)+F(x_0-2h)}{4h^2}=A_0+\sum_{n=1}^{\infty}A_n\left(\frac{\sin nh}{nh}\right)^2,$$

where A_0 stands for $\frac{1}{2}a_0$ and A_n for $a_n \cos nx_0+\beta_n \sin nx_0$.

Now, as $\sum_{0}^{\infty}A_n$ is, by hypothesis, convergent, let

$$A_0+A_1+\ldots+A_{n-1}=f(x_0)+\epsilon_n,$$

where, corresponding to any quantity $\delta>0$, however small, it is always possible to find a number m such that, for $n>m$, $|\epsilon_n|<\delta$.

Let us take now h to be sufficiently small so that $mh<\pi$ and using

$$\Delta A_n=\epsilon_{n+1}-\epsilon_n,$$

change the series

$$\sum_{n=0}^{\infty}A_n\left(\frac{\sin nh}{nh}\right)^2$$

into

$$f(x_0)+\sum_{n=1}^{\infty}\epsilon_n\left[\left\{\frac{\sin(n-1)h}{(n-1)h}\right\}^2-\left(\frac{\sin nh}{nh}\right)^2\right].$$

(*b*) Now divide this series into three parts, bringing together

(*i*) all the terms corresponding to $n=1, 2,\ldots, m$;

(*ii*) all the terms corresponding to the values of n from $m+1$ to s, where s is the greatest integer in $\frac{\pi}{h}$;

(*iii*) all the terms corresponding to the values of n from $s+1$ to ∞. The first part consists of terms varying in a continuous manner and can, consequently, be made as nearly equal to its limit zero as we please, by choosing h to be sufficiently small.

In the second part, the factor of ϵ_n is always positive, and therefore this part is numerically less than

$$\delta\left\{\left(\frac{\sin mh}{mh}\right)^2-\left(\frac{\sin sh}{sh}\right)^2\right\}.$$

Dividing the general term in the third part into two parts as in

$$\epsilon_n\left[\left\{\frac{\sin(n-1)h}{(n-1)h}\right\}^2-\left\{\frac{\sin(n-1)h}{nh}\right\}^2\right]$$

$$+\epsilon_n\left[\left\{\frac{\sin(n-1)h}{nh}\right\}^2-\left(\frac{\sin nh}{nh}\right)^2\right],$$

it is seen to be numerically less than

$$\delta\left\{\frac{1}{(n-1)^2h^2}-\frac{1}{n^2h^2}\right\}+\frac{\delta}{n^2h}.$$

Hence the third part is numerically less than

$$\delta\left(\frac{1}{s^2h^2}+\frac{1}{sh}\right)<\delta\left(\frac{1}{\pi^2}+\frac{1}{\pi}\right).$$

Therefore, it is proved that, with decreasing h, the sum of the three parts remains less than

$$\delta\left(1+\frac{1}{\pi}+\frac{1}{\pi^2}\right),$$

and is, consequently, 0. Hence

$$\operatorname{Lim}_{h=0}\frac{F(x_0+2h)-2F(x_0)+F(x_0-2h)}{4h^2}=f(x_0),$$

APPENDIX B.

Notes, additions and corrections.

Page 3. Line 14 of the foot-note.

For "representabilit" *read* "representability."

Page 4. Line 4 from the bottom.

For "$o'h)$" *read* "$O(h)$."

Page 5. Lines 19, 21. If $f_1(t)$, $f_2(t)$ be two functions which are monotone in the neighbourhood of $t=t_0$, then it is said that $f_1 \succ f_2$ if $\lim_{t=t_0} \frac{f_1}{f_2} = \infty$; similarly $f_1 \prec f_2$ if $\lim_{t=t_0} \frac{f_1}{f_2} = 0$.

Page 24. Last line. *For* "$\frac{\ }{2}$" *read* "$\frac{1}{2}$".

Pages 27-31. *For* "v" *read* "ν."

Page 36. Line 2. *For* "$G=O(x)$" *read* "$G=o(x)$."

Page 38. Line 1. *For* "$\cos \frac{1}{\theta}$ or $\sin \frac{1}{\theta}$" *read* "$\cos \log \frac{1}{\theta}$ or $\sin \log \frac{1}{\theta}$"

Page 38. Line 5. *For* "21" *read* "22."

Pages 39, 51, 53, 54. Wherever "v" occurs it should be taken to mean "ν."

Page 41. Last line of the foot-note.

For "$^r\pi$" *read* "π."

Page 48. Foot-note. The reference is to the paper quoted in Art. 7.

Page 49. Line 1. *For* "$pv)$" *read* "$p(v)$."

Page 55. Foot-note. See Art. 7 for the paper quoted.

Page 61. The proof given by Lebesgue in his "Lecons," pp. 94-96, is *vitiated* by an oversight on his part and has not been reproduced

here; on p. 95 in the 7th line the factor outside the square brackets is $1/n\pi \sin^2 \frac{\pi}{2n+1}$ and not $1/n\pi \sin \frac{\pi}{2n+1}$ as stated by Lebesgue. The subsequent reasoning fails with the correct value of the factor.

Page 63. Line 7. *For* "$\frac{n-k+1}{n-k+1}$" *read* "$\frac{1}{k-n+1}$".

Page 73. Line 6. *For* "propery" *read* "property."

For the most recent generalization of this fourth property, see G. Prasad's paper, "On the summability (C1) of the derived series of the Fourier series of an indefinite integral at a point where the integrand has a discontinuity of the second kind." (*Bulletin of the Calcutta Mathematical Society*, Vol. XIX, pp. 96-100.)

Page 81. Line 4. *For* "so that" *read* "so chosen that."

Page 82. Line 8 of the first foot-note.
For "A_n^r" *read* "A_n^r."

Page 85. Line 1. *For* "43" *read* "43a."

Page 86. Foot-note. *For* "$(f_1, f_1), (f_2, f_2)$"
read "$(f_1, f_1), (f_2, f_2)$."

Page 87. Line 4. *For* "result" *read* "equation (1)."

Page 90. Line 5. *For* "δ" *read* "$1-r$."

Page 92. Line 12. *For* "$n^{+\frac{1}{2p}}$" *read* "$n^{1+\frac{1}{2p}}$."

Page 96. Line 5. *For* "$-\frac{1}{2p-1}$" *read* "$-\frac{1}{2p}-1$."

Page 102. Line 4. *For* "n_1 or n_2" *read* "n_1 and n_2."

Page 102. In his paper, "On the strong summability (C1) of the Fourier series of a function at a point where the function has an infinite

discontinuity of the second kind" (*Bulletin of the Calcutta Mathematical Society*, Vol. XIX, pp. 127-134), G. Prasad has proved the following general result: If $\psi \succ \log \frac{1}{t^2}$, $\frac{\chi}{\psi'} \prec t$ and, further,

$$\int_0^t \{\chi'(u)\}^2 du \prec t^{1-\lambda}, \text{ where } 1 > \lambda > 0,$$

then

$$\sum_0^n |s_m - s|^q = O(n)$$

for every positive q.

Page 128. Line 17. *For* "Menchof" *read* "Menchoff."

Page 131. In a paper, to be published in the *Bulletin of the Calcutta Mathematical Society*, Mr. H. P. Banerji has *directly verified* the ordinary summability (C1) as well as the strong summability (C1) at $\theta=0$ of the series considered in (*c*).

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